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Further q-congruences on double basic hypergeometric sums

by

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Abstract

In this paper, we establish a q-congruence which unifies one kind of q-congruences on double sums and the result confirms Wei and Li's conjectures as well. Meanwhile, we provide one new q-supercongruence on double sums.

Key Words: Basic hypergeometric series, q-congruences, creative microscoping method, cyclotomic polynomials.

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1 Introduction

In 2011, Long [11] conjectured that for any odd prime p,

$$\sum_{k=0}^{(p-1)/2}(-1)^k\frac{6k+1}{8^k}\frac{(1/2)_k^3}{k!^3}\sum_{j=1}^{k}\left\{\frac{1}{(2j-1)^2}-\frac{1}{16j^2}\right\}\equiv 0 \pmod{p}, \tag{1}$$

where the Pochhammer symbol is given by

$$(a)_0=1 \qquad \text{and} \qquad (a)_n=a(a+1)\cdots(a+n-1) \quad \text{for} \quad n\in\mathbb{Z}^+.$$

In the last few years, this interesting congruence involving double series attracted many researchers' attentions. In 2015, Swisher [13] confirmed Long's conjecture (1). Later, Gu and Guo[4] established a beautiful q-analogue of (1): for any positive odd integer n,

$$\sum_{k=0}^{(n-1)/2}(-1)^k[6k+1]\frac{(q;q^2)_k^3}{(q^4;q^4)_k^3}\sum_{j=1}^{k}\left\{\frac{q^{2j-1}}{[2j-1]^2}-\frac{q^{4j}}{[4j]^2}\right\}\equiv 0 \pmod{\Phi_n(q)}. \tag{2}$$

Here, the *q-shifted factorial* is defined as $(a;q)_0=1$, $(a;q)_n=(1-a)(1-aq)\cdots(1-aq^{n-1})$ for $n\in\mathbb{Z}^+$ and $[n]=[n]_q=1+q+\cdots+q^{n-1}$ is the *q-integer*. $\Phi_n(q)$ stands for the n-th *cyclotomic polynomial* in q:

$$\Phi_n(q)=\prod_{\substack{1\leqslant k\leqslant n\\ \gcd(n,k)=1}}(q-\zeta^k),$$

with ζ an n-th primitive root of unity.

On the other hand, in 1997, Van Hamme [14] conjectured 13 congruences, whose q-analogues have been investigated by many authors. More progress on q-congruences can be found in [15, 8].

Recently, Wang and Yu [16] and Wei and Li [17] presented some q-congruences on double basic hypergeometric sums similar to (2). In particular, Wei and Li [17, Theorem 1] proved that

$$\sum_{k=0}^{(n-r)/d}[2dk+r]\frac{(q^r;q^d)_k^3(xq^r,yq^r,zq^r;q^d)_k}{(q^d;q^d)_k^3(q^d/x,q^d/y,q^d/z;q^d)_k}\frac{q^{(2d-3r)k}}{(xyz)^k} \times\sum_{j=1}^{k}\left(\frac{q^{dj-d+r}}{[dj-d+r]^2}-\frac{q^{dj}}{[dj]^2}\right)\equiv 0 \pmod{\Phi_n(q)}, \tag{3}$$

where $n>1,d>1,r>0$ are integers with $(r,d)=1$ and $n\equiv r \pmod d$ such that $n\geq r$. In the same paper, Wei and Li also proposed the following conjectures[17, Conjecture 1 and 2].

Conjecture 1.1. *Let $n>1,d>1,r>0,m\geq 0$ be integers with $(r,d)=1$ and $n\equiv r \pmod d$ such that $n\geq r$. Then modulo $\Phi_n(q)$,*

$$\sum_{k=0}^{(n-r)/d}(-1)^k[2dk+r]\frac{(q^r;q^d)_k^{2m+1}}{(q^d;q^d)_k^{2m+1}}q^{d\binom{k}{2}+m(d-r)k}\sum_{j=1}^{k}\left(\frac{q^{dj-d+r}}{[dj-d+r]^2}-\frac{q^{dj}}{[dj]^2}\right)\equiv 0.$$

Conjecture 1.2. *Let $n>1,d>1,r>0,m\geq 0$ be integers with $(r,d)=1$ and $n\equiv r \pmod d$ such that $n\geq r$. Then modulo $\Phi_n(q)$,*

$$\sum_{k=0}^{(n-r)/d}[2dk+r]\frac{(q^r;q^d)_k^{2m+2}}{(q^d;q^d)_k^{2m+2}}q^{m(d-r)k-rk}\sum_{j=1}^{k}\left(\frac{q^{dj-d+r}}{[dj-d+r]^2}-\frac{q^{dj}}{[dj]^2}\right)\equiv 0.$$

Inspired by the work just mentioned, we shall prove these two conjectures in this paper. In fact, the $m=0$ cases of both conjectures can be obtained by fixing $(x,y,z)=(q^{d-r},q^{d-r},\infty)$ and $(q^{d-r},q^{d-r},1)$ in (3), respectively. And the $m\geq 1$ cases rely on the following theorem, which can be deemed as a generalization of both conjectures.

Theorem 1.3. *Let $n>1,d>1,m\geq 1$ and r be integers with $(r,d)=(n,d)=1$, $tn+d-dn\leq r\leq n$, where t is the least positive integer satisfying $tn\equiv r \pmod d$. Let b_i,c_j be indeterminates, where $1\leq i\leq m, 1\leq j\leq m-1$. Then*

$$\sum_{k=0}^{M}[2dk+r]\frac{(b_1,c_1,\ldots,b_{m-1},c_{m-1},b_m;q^d)_k(q^r;q^d)_k^3}{(q^{d+r}/b_1,q^{d+r}/c_1,\ldots,q^{d+r}/b_{m-1},q^{d+r}/c_{m-1},q^{d+r}/b_m)_k(q^d;q^d)_k^3} \times\left(\frac{q^{mr+md-2r}}{b_1c_1\cdots b_{m-1}c_{m-1}b_m}\right)^k\times\sum_{j=1}^{k}\left(\frac{q^{dj-d+r}}{[dj-d+r]^2}-\frac{q^{dj}}{[dj]^2}\right)\equiv 0 \pmod{\Phi_n(q)},$$

where $M=(tn-r)/d$ or $n-1$.

Obviously, taking $m=2, t=1, b_1=xq^r, c_1=yq^r, b_2=zq^r$ in Theorem 1.3, we obtain (3). The $m\ge 1$ cases of Conjecture 1.1 and Conjecture 1.2 can be derived by choosing $(t, b_1, c_1, \dots, b_{m-1}, c_{m-1}, b_m) \to (1, q^r, q^r, \dots, q^r, q^r, \infty)$ and $(1, q^r, q^r, \dots, q^r, q^r, q^r)$ in Theorem 1.3 respectively.

In fact, with the change of the parameters and the indeterminates, Theorem 1.3 can produce a huge amount of q-congruences on double series. In the last few years, with the help of transformation formulas for basic hypergeometric series, such as Watson's ${}_8\phi_7$-transformation formula [2, Appendix (III.18)], a lot of q-congruences and q-supercongruences were proved by the 'creative microscoping' method. Almost all of them, including Van Hamme's supercongruences, can produce corresponding q-congruences involving double series by Theorem 1.3.

Some special cases of Theorem 1.3 can be further strengthened to q-supercongruences modulo the square of a cyclotomic polynomial. For example, Song and Wang [12] proved a new q-supercongruence related to the supercongruence (D.2) of Van Hamme. Fang and Guo [3] investigated the double sums related to Van Hamme's supercongruences (A.2) and (H.2). Guo and Lian [5] presented the q-supercongruences related to Van Hamme's supercongruences (C.2), and (J.2).

Here, we shall present the q-supercongruence on double sums corresponding to Van Hamme's supercongruences (G.2).

Theorem 1.4. *Let n be a positive integer with $n\equiv 1 \pmod 4$. Then, modulo $\Phi_n(q)^2$,*

$$\sum_{k=0}^{B}[8k+1]\frac{(q;q^4)_k^4}{(q^4;q^4)_k^4}q^{2k}\sum_{j=1}^{k}\left(\frac{q^{4j-3}}{[4j-3]^2}-\frac{q^{4j}}{[4j]^2}\right)$$
$$\equiv q^{(1-n)/4}[n]\frac{(q^2;q^4)_{(n-1)/4}}{(q^4;q^4)_{(n-1)/4}}\left(\frac{(n^2-1)(1-q)^2}{24}+\sum_{k=1}^{(n-1)/4}\frac{q^{4k-2}}{[4k-2]^2}\right),$$

where $B=(n-1)/4$ or $n-1$.

The rest of our paper is arranged as follows. The proof of Theorem 1.3 will be shown in Section 2. Theorem 1.4 will be proven in Section 3. In the last section, we give one more such q-supercongruence.

2 Proof of Theorem 1.3

In the proof, we shall make full use of the sum of a powerful transformation formula due to Andrews[1, Theorem 4], which can be stated as follows:

$$\begin{aligned}
&\sum_{k\ge 0}\frac{(a,q\sqrt{a},-q\sqrt{a},b_1,c_1,\ldots,b_m,c_m,q^{-N};q)_k}{(q,\sqrt{a},-\sqrt{a},aq/b_1,aq/c_1,\ldots,aq/b_m,aq/c_m,aq^{N+1};q)_k}\left(\frac{a^mq^{m+N}}{b_1c_1\cdots b_mc_m}\right)^k\\
&\quad=\frac{(aq,aq/b_mc_m;q)_N}{(aq/b_m,aq/c_m;q)_N}\sum_{j_1,\ldots,j_{m-1}\ge 0}\frac{(aq/b_1c_1;q)_{j_1}\cdots(aq/b_{m-1}c_{m-1};q)_{j_{m-1}}}{(q;q)_{j_1}\cdots(q;q)_{j_{m-1}}}\\
&\qquad\times\frac{(b_2,c_2;q)_{j_1}\cdots(b_m,c_m;q)_{j_1+\cdots+j_{m-1}}}{(aq/b_1,aq/c_1;q)_{j_1}\cdots(aq/b_{m-1},aq/c_{m-1};q)_{j_1+\cdots+j_{m-1}}}\\
&\qquad\times\frac{(q^{-N};q)_{j_1+\cdots+j_{m-1}}}{(b_mc_mq^{-N}/a;q)_{j_1+\cdots+j_{m-1}}}\frac{(aq)^{j_{m-2}+\cdots+(m-2)j_1}q^{j_1+\cdots+j_{m-1}}}{(b_2c_2)^{j_1}\cdots(b_{m-1}c_{m-1})^{j_1+\cdots+j_{m-2}}}.
\end{aligned}\tag{4}$$

In order to prove Theorem 1.3, we first need the following congruence, which is a generalization of Liu and Wang's result[9, Lemma 2].

Lemma 2.1. *Let $n>1,d>1,m\ge 1$ and r be integers with $(r,d)=(n,d)=1$, $tn+d-dn\le r\le n$, where t is the least positive integer satisfying $tn\equiv r \pmod d$. Let b_i,c_j be indeterminates, where $1\le i\le m, 1\le j\le m-1$. Then, modulo $\Phi_n(q)(1-aq^{tn})(a-q^{tn})$, we have*

$$\begin{aligned}
&\sum_{k=0}^{M}[2dk+r]\frac{(b_1,c_1,\ldots,b_{m-1},c_{m-1},b_m;q^d)_k(q^r,aq^r,q^r/a;q^d)_k}{(q^{d+r}/b_1,q^{d+r}/c_1,\ldots,q^{d+r}/b_{m-1},q^{d+r}/c_{m-1},q^{d+r}/b_m)_k(q^d,aq^d,q^d/a;q^d)_k}\\
&\quad\times\left(\frac{q^{mr+md+tn-n-2r}}{b_1c_1\cdots b_{m-1}c_{m-1}b_m}\right)^k\equiv[tn]\frac{(b_mq^r;q^d)_{(tn-r)/d}}{(q^{d+r}/b_m;q^d)_{(tn-r)/d}}(b_m)^{(r-tn)/d}\\
&\quad\times\sum_{j_1,\ldots,j_{m-1}\ge 0}\frac{(q^{d+r}/b_1c_1;q^d)_{j_1}\cdots(q^{d+r}/b_{m-1}c_{m-1};q^d)_{j_{m-1}}}{(q^d;q^d)_{j_1}\cdots(q^d;q^d)_{j_{m-1}}}\\
&\qquad\times\frac{(b_2,c_2;q^d)_{j_1}\cdots(b_m,aq^r;q^d)_{j_1+\cdots+j_{m-1}}}{(q^{d+r}/b_1,q^{d+r}/c_1;q^d)_{j_1}\cdots(q^{d+r}/b_{m-1},q^{d+r}/c_{m-1};q^d)_{j_1+\cdots+j_{m-1}}}\\
&\qquad\times\frac{(q^r/a;q^d)_{j_1+\cdots+j_{m-1}}}{(b_mq^r;q^d)_{j_1+\cdots+j_{m-1}}}\frac{q^{(d+r)(j_{m-2}+\cdots+(m-2)j_1)+d(j_1+\cdots+j_{m-1})}}{(b_2c_2)^{j_1}\cdots(b_{m-1}c_{m-1})^{j_1+\cdots+j_{m-2}}},
\end{aligned}\tag{5}$$

where $M=(tn-r)/d$ or $n-1$.

Proof. When $a = q^{tn}$ or $a = q^{-tn}$, in terms of (4), we can catch hold of

$$\sum_{k=0}^{(tn-r)/d} \frac{[2dk+r]\,(b_1, c_1, \ldots, b_{m-1}, c_{m-1}, b_m; q^d)_k (q^r, q^{r+tn}, q^{r-tn}; q^d)_k}{(q^{d+r}/b_1, q^{d+r}/c_1, \ldots, q^{d+r}/b_{m-1}, q^{d+r}/c_{m-1}, q^{d+r}/b_m)_k (q^d, q^{d+tn}, q^{d-tn}; q^d)_k}$$
$$\times \left(\frac{q^{mr+md+tn-n-2r}}{b_1 c_1 \cdots b_{m-1} c_{m-1} b_m} \right)^k = [tn] \frac{(b_m q^r; q^d)_{(tn-r)/d}}{(q^{d+r}/b_m; q^d)_{(tn-r)/d}} (b_m)^{(r-tn)/d}$$
$$\times \sum_{j_1, \ldots, j_{m-1} \geq 0} \frac{(q^{d+r}/b_1 c_1; q^d)_{j_1} \cdots (q^{d+r}/b_{m-1} c_{m-1}; q^d)_{j_{m-1}}}{(q^d; q^d)_{j_1} \cdots (q^d; q^d)_{j_{m-1}}}$$
$$\times \frac{(b_2, c_2; q^d)_{j_1} \cdots (b_m, q^{r+tn}; q^d)_{j_1 + \cdots + j_{m-1}}}{(q^{d+r}/b_1, q^{d+r}/c_1; q^d)_{j_1} \cdots (q^{d+r}/b_{m-1}, q^{d+r}/c_{m-1}; q^d)_{j_1 + \cdots + j_{m-1}}}$$
$$\times \frac{(q^{r-tn}; q^d)_{j_1 + \cdots + j_{m-1}}}{(b_m q^r; q^d)_{j_1 + \cdots + j_{m-1}}} \frac{q^{(d+r)(j_{m-2} + \cdots + (m-2) j_1) + d(j_1 + \cdots + j_{m-1})}}{(b_2 c_2)^{j_1} \cdots (b_{m-1} c_{m-1})^{j_1 + \cdots + j_{m-2}}}. \tag{6}$$

This means (5) holds modulo $(1 - aq^n)(a - q^n)$ for $M = (tn - r)/d$. With the help of the relation $(q^{r-tn}; q^d)_k = 0$ for any k with $(tn - r)/d < k < n$, we get that (5) also holds modulo $(1 - aq^{tn})(a - q^{tn})$ for $M = n - 1$.

By utilizing Lemma [7, Lemma 2.1], we can easily check that the k-th term and the $((tn - r)/d - k)$-th term on the left-hand side of (5) can cancel each other modulo $\Phi_n(q)$. Therefore, the left-hand side of (5) is congruent to 0 modulo $\Phi_n(q)$ for $M = (tn - r)/d$. Moreover, since $(q^r; q^d)_k/(q^d; q^d)_k \equiv 0 \pmod{\Phi_n(q)}$ for $(tn - r)/d < k < n$, we can immediately prove (5) is true modulo $\Phi_n(q)$.

Since $\Phi_n(q), (1 - aq^{tn}), (a - q^{tn})$ are pairwise coprime polynomials in q, we finish the proof. □

Proof of Theorem 1.3. We shall pay attention to calculate the following term, and denote it as $S(q)$.

$$S(q) = \sum_{k=0}^{M} [2dk+r] \frac{(b_1, c_1, \ldots, b_{m-1}, c_{m-1}, b_m; q^d)_k (q^r; q^d)_k^3}{(q^{d+r}/b_1, q^{d+r}/c_1, \ldots, q^{d+r}/b_{m-1}, q^{d+r}/c_{m-1}, q^{d+r}/b_m)_k (q^d; q^d)_k^3}$$
$$\times \left(\frac{q^{mr+md+tn-n-2r}}{b_1 c_1 \cdots b_{m-1} c_{m-1} b_m} \right)^k$$
$$- \sum_{k=0}^{M} [2dk+r] \frac{(b_1, c_1, \ldots, b_{m-1}, c_{m-1}, b_m; q^d)_k (q^r, q^{r+tn}, q^{r-tn}; q^d)_k}{(q^{d+r}/b_1, q^{d+r}/c_1, \ldots, q^{d+r}/b_{m-1}, q^{d+r}/c_{m-1}, q^{d+r}/b_m)_k (q^d, q^{d+tn}, q^{d-tn}; q^d)_k}$$
$$\times \left(\frac{q^{mr+md+tn-n-2r}}{b_1 c_1 \cdots b_{m-1} c_{m-1} b_m} \right)^k.$$

On one hand, $S(q)$ can be simplified like this,

$$S(q)=\sum_{k=0}^{M}[2dk+r]\frac{(b_1,c_1,\dots,b_{m-1},c_{m-1},b_m;q^d)_k(q^r;q^d)_k}{(q^{d+r}/b_1,q^{d+r}/c_1,\dots,q^{d+r}/b_{m-1},q^{d+r}/c_{m-1},q^{d+r}/b_m)_k(q^d;q^d)_k}\\ \times\left(\frac{q^{mr+md+tn-n-2r}}{b_1c_1\cdots b_{m-1}c_{m-1}b_m}\right)^k\\ \times\frac{(q^r,q^r,q^{d+tn},q^{d-tn};q^d)_k-(q^{r+tn},q^{r-tn},q^d,q^d;q^d)_k}{(q^d,q^d,q^{d+tn},q^{d-tn};q^d)_k}. \tag{7}$$

Noticing that $q^n\equiv 1 \pmod{\Phi_n(q)}$, we obtain

$$\begin{aligned}(q^{d+tn},q^{d-tn};q^d)_k&=\prod_{j=1}^{k}(1-q^{dj+tn})(1-q^{dj-tn})\\ &=\prod_{j=1}^{k}((1-q^{dj})^2-(1-q^{tn})^2q^{dj-tn})\\ &\equiv (q^d;q^d)_k^2-(q^d;q^d)_k^2\sum_{j=1}^{k}\frac{(1-q^{tn})^2}{(1-q^{dj})^2}q^{dj-tn} \pmod{\Phi_n(q)^4}.\end{aligned}$$

Similarly, we can get

$$(q^{r+tn},q^{r-tn};q^d)_k\equiv (q^r;q^d)_k^2-(q^r;q^d)_k^2\sum_{j=1}^{k}\frac{(1-q^{tn})^2}{(1-q^{dj-d+r})^2}q^{dj-d+r-tn} \pmod{\Phi_n(q)^4}.$$

Combining the above two equations, we are led to

$$\begin{aligned}&(q^r,q^r,q^{d+tn},q^{d-tn};q^d)_k-(q^{r+tn},q^{r-tn},q^d,q^d;q^d)_k\\ &\equiv (q^r,q^r,q^d,q^d;q^d)_k[tn]^2\sum_{j=1}^{k}\left(\frac{q^{dj-d+r-tn}}{[dj-d+r]^2}-\frac{q^{dj-tn}}{[dj]^2}\right) \pmod{\Phi_n(q)^4}.\end{aligned} \tag{8}$$

Applying the relation (8) in (7), we obtain

$$S(q)\equiv\sum_{k=0}^{M}[2dk+r]\frac{(b_1,c_1,\dots,b_{m-1},c_{m-1},b_m;q^d)_k(q^r;q^d)_k}{(q^{d+r}/b_1,q^{d+r}/c_1,\dots,q^{d+r}/b_{m-1},q^{d+r}/c_{m-1},q^{d+r}/b_m)_k(q^d;q^d)_k}\\ \times\left(\frac{q^{mr+md+tn-n-2r}}{b_1c_1\cdots b_{m-1}c_{m-1}b_m}\right)^k\\ \times\frac{(q^r,q^r;q^d)_k}{(q^{d+tn},q^{d-tn};q^d)_k}[tn]^2\sum_{j=1}^{k}\left(\frac{q^{dj-d+r-tn}}{[dj-d+r]^2}-\frac{q^{dj-tn}}{[dj]^2}\right) \pmod{\Phi_n(q)^4}. \tag{9}$$

On the other hand, substituting the $a=1$ case of Lemma 2.1 into the first sum in $S(q)$, and utilizing (6) in the second sum in $S(q)$, we obtain

$$S(q)\equiv 0 \pmod{\Phi_n(q)^3}, \tag{10}$$

due to the relation $(q^{r+tn}, q^{r-tn}; q^d)_{j_1+\cdots+j_{m-1}} \equiv (q^r, q^r; q^d)_{j_1+\cdots+j_{m-1}} \pmod{\Phi_n(q)^2}$.

Combining (7) and (10) together and noticing $q^n \equiv 1 \pmod{\Phi_n(q)}$, we finish proving Theorem 1.3. □

3 Proof of Theorem 1.4

Proof of Theorem 1.4. The proof is very similar to that of Theorem 1.3. Denoting $U(q)$ as follows,

$$U(q) = \sum_{k=0}^{B} [8k+1] \frac{(q;q^4)_k^4}{(q^4;q^4)_k^4} q^{2k} - \sum_{k=0}^{B} [8k+1] \frac{(q^{1+n};q^4)_k (q^{1-n};q^4)_k (q;q^4)_k^2}{(q^{4+n};q^4)_k (q^{4-n};q^4)_k (q^4;q^4)_k^2}, \tag{11}$$

we obtain

$$U(q) \equiv \sum_{k=0}^{B} [8k+1] \frac{(q;q^4)_k^4}{(q^4;q^4)_k^4} q^{2k} [n]^2 \sum_{j=1}^{k} \left(\frac{q^{4j-3}}{[4j-3]^2} - \frac{q^{4j}}{[4j]^2} \right) \pmod{\Phi_n(q)^4}. \tag{12}$$

On the other hand, the $a = q^n$ case of [10, Theorem 7] reads

$$\sum_{k=0}^{B} [8k+1] \frac{(q^{1+n};q^4)_k (q^{1-n};q^4)_k (q;q^4)_k^2}{(q^{4+n};q^4)_k (q^{4-n};q^4)_k (q^4;q^4)_k^2} = q^{(1-n)/4} [n] \frac{(q^2;q^4)_{(n-1)/4}}{(q^4;q^4)_{(n-1)/4}}.$$

Substituting the above equation, [10, Theorem 1] and (12) into (11), following the same path in the proof of Theorem 1.3, we can prove Theorem 1.4 immediately. □

4 More q-supercongruences on double sums

Theorem 4.1. *Let d, r, n be integers satisfying $d \geq 3, r \leq d-2$, and $n \geq d-r$, such that d and r are coprime, and $n \equiv -r \pmod{d}$. Then*

$$\sum_{k=0}^{C} [2dk+r] \frac{(q^r;q^d)_k^{2d}}{(q^d;q^d)_k^{2d}} q^{d(d-1-r)k} \sum_{j=1}^{k} \left(\frac{q^{dj-d+r}}{[dj-d+r]^2} - \frac{q^{dj}}{[dj]^2} \right) \equiv 0 \pmod{\Phi_n(q)^2}, \tag{13}$$

where $C = (dn-n-r)/d$ or $n-1$.

We first list the following lemma in order to prove Theorem 4.1.

Lemma 4.2 (Guo and Schlosser[6])**.** *Let d, r, n be integers satisfying $d \geq 3, r \leq d-2$, and $n \geq d-r$, such that d and r are coprime, and $n \equiv -r \pmod{d}$. Then*

$$\sum_{k=0}^{n-1} [2dk+r] \frac{(q^r;q^d)_k^{2d}}{(q^d;q^d)_k^{2d}} q^{d(d-1-r)k} \equiv 0 \pmod{\Phi_n(q)^4}. \tag{14}$$

Proof of Theorem 4.1. We denote $R(q)$ as follows,

$$R(q) = \sum_{k=0}^{C} [2dk+r] \frac{(q^r;q^d)_k^{2d}}{(q^d;q^d)_k^{2d}} q^{d(d-1-r)k} - \sum_{k=0}^{C} [2dk+r] \frac{(q^r;q^d)_k^{2d-2}(q^{r+tn},q^{r-tn};q^d)_k}{(q^d;q^d)_k^{2d-2}(q^{d+tn},q^{d-tn};q^d)_k} q^{d(d-1-r)k},$$

where $t = d-1$.

Noticing the relation (8), we obtain

$$R(q) \equiv \sum_{k=0}^{C} [2dk+r] \frac{(q^r;q^d)_k^{2d}}{(q^d;q^d)_k^{2d}} q^{d(d-1-r)k} [tn]^2 \sum_{j=1}^{k} \left(\frac{q^{dj-d+r}}{[dj-d+r]^2} - \frac{q^{dj}}{[dj]^2} \right) \pmod{\Phi_n(q)^4}.$$

On the other hand, Lemma 4.2 implies the first sum in $R(q)$ is congruent to 0 modulo $\Phi_n(q)^4$, because (14) also holds when the left-hand side truncates at $(dn-n-r)/d$. The second sum in $R(q)$ is congruent to 0 modulo $\Phi_n(q)^4$ as well, this can be proven by following the same path in the proof of Lemma 4.2. Therefore, $R(q) \equiv 0 \pmod{\Phi_n(q)^4}$. The proof is finished. □

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