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rq-Convexity of lattice graphs

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Abstract

Let $\{x, y, w, z\} \subset \mathbb{R}^d$. If $\operatorname{conv}\{x, y, w, z\}$ is a non-degenerate rectangle, then we call the set $\{x, y, w, z\}$ a *rectangular quadruple*. Let $M \subset \mathbb{R}^d$ with $\operatorname{card}M \geq 4$. If, for any $x, y \in M$, there exists a rectangular quadruple $\{x, y, w, z\} \subset M$, we say that M is *rq-convex* and the pair x, y have the *rq-property* in M. In this paper, we consider rq-convexity of lattice graphs which are in the planar square and triangular lattices and the cubic lattice in 3-space.

Key Words: Rectangular quadruple, rq-convexity, lattice graphs, cubic lattice.
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1 Introduction

In 1974, the third author proposed at the meeting on Convexity in Oberwolfach the investigation of the following general convexity concept. Let $\mathcal{F}$ be a family of sets in $\mathbb{R}^d$ (always $d \geq 2$). A set $M \subset \mathbb{R}^d$ is called $\mathcal{F}$*-convex*, if for any pair of distinct points $x, y \in M$, there is a set $F \in \mathcal{F}$, such that $x, y \in F$ and $F \subset M$ [1].

Blind, Valette and the third author [1], and also Böröczky, Jr. [2] investigated the rectangular convexity, the case when $\mathcal{F}$ is the family of all non-degenerate rectangles. Magazanik and Perles [4] studied staircase connectedness. The third author [10] introduced the right convexity. Then the second and the third author [9] [8] investigated the right triple convexity. Li and the last two authors [3] dealt with the right quadruple convexity, abbreviated as rq-convexity. Wang, Nie and the last two authors studied the poidge-convexity and the thin right triangle convexity (see [7], [5]). All these concepts are particular cases of $\mathcal{F}$-convexity.

This paper is about the rq-convexity in lattice graphs. The lattices which will be considered are the planar square and triangular lattices and the cubic lattice in 3-space.

2 Definitions

For a set $M \subset \mathbb{R}^d$, we denote by $\operatorname{conv}M$ its convex hull, by $\overline{M}$ its affine hull and by $\operatorname{cl}M, \operatorname{int}M, \operatorname{bd}M$ its closure, relative interior and relative boundary, which means in the topology of $\overline{M}$.

Put $x_1x_2\ldots x_n = \operatorname{conv}\{x_1, x_2, \ldots, x_n\}$, for $x_1, \ldots, x_n \in \mathbb{R}^d$. Thus, for distinct points $x, y \in \mathbb{R}^d$, xy denotes the line-segment from x to y, and $\overline{xy}$ the line through x, y; let H_{xy} be the hyperplane through x orthogonal to $\overline{xy}$; the hypersphere with a diameter xy is denoted by C_{xy}.

A *diameter* of a closed set $M \subset \mathbb{R}^d$ is a line-segment ab such that $\|a-b\| = \sup\{\|x-y\| : x, y \in M\}$ and $a, b \in M$. We write $\mathrm{diam} M = \|a-b\|$.

For any two sets $H_1, H_2 \subset \mathbb{R}^d$, $H_1 \parallel H_2$ means that $\overline{H_1}$ is parallel to $\overline{H_2}$, and $H_1 \perp H_2$ means that $\overline{H_1}$ and $\overline{H_2}$ are orthogonal.

A set of four points $\{w, x, y, z\} \subset \mathbb{R}^d$ is called a *rectangular quadruple*, if $wxyz$ is a non-degenerate rectangle.

Let $M \subset \mathbb{R}^d$ with $\mathrm{card} M \geq 4$. If, for $x, y \in M$, there exists a rectangular quadruple $\{w, x, y, z\} \subset M$, we say that x, y have the *rq-property* in M. If any pair of points in M have the rq-property, then we call the set M *rq-convex*.

If there exists a point $k \in M$ such that for any $x \in M$, k, x enjoy the rq-property in M, then M is an *rq-starshaped set*. The set of points in M which can play the role of k form the *kernel* of M.

3 rq-Convexity of square lattice graphs

Consider the norm $\|(q_1, , q_2, \ldots, q_d)\|_m = \max\{|q_1|, |q_2|, \ldots, |q_d|\}$, defining in $\mathbb{Z}^d$ the discs of radius $n \in \mathbb{N}$

$$Q(n) = \{(x_1, x_2, \ldots, x_d) \in \mathbb{Z}^d : \|(x_1, x_2, \ldots, x_d)\|_m \leq n\},$$

centred at the origin $\mathbf{0}$. For $d = 2$, Li, Yuan and Zamfirescu [3] proved that, besides $Q(n)$, the set $Q(n) \setminus \{\mathbf{0}\}$ is rq-convex, while $Q(n) \setminus Q(n-2)$ and $Q(n) \setminus Q(n-1)$ ($n \geq 3$) are not.

We remark that, however, for any $1 \leq i \leq n-1$, the set $Q(n) \setminus Q(i)$ is rq-starshaped. The four points $(n, n), (n, -n), (-n, n), (-n, -n)$, are in the kernel.

Now, we consider subsets of $Q(n)$, for $d \geq 3$.

Theorem 1. *For any $0 \leq i \leq n-1$, the set $Q(n) \setminus Q(i)$ in $\mathbb{Z}^d$ $(d \geq 3)$ is rq-convex.*

Proof. Let $x = (x_1, x_2, \ldots, x_d), y = (y_1, y_2, \ldots, y_d)$ be two points in $Q = Q(n) \setminus Q(i)$.

Case 1. $x_1 \neq y_1$ and $x_j = y_j$ $(j = 2, \ldots, d)$.

Choose $s = (x_1, s_2, \ldots, s_d), t = (y_1, s_2, \ldots, s_d) \in Q$ such that $(s_2, \ldots, s_d) \neq (x_2, \ldots, x_d)$.

Case 2. $x_1 \neq y_1$ and $x_2 \neq y_2$.

For some $j \in \{1, 2\}$, if $|x_j|, |y_j| \leq i$ or $|x_j|, |y_j| > i$, then take $s = (x_1, y_j, x_3, \ldots, x_d), t = (y_1, x_j, y_3, \ldots, y_d)$.

If $|x_1|, |y_2| \leq i$ and $|x_2|, |y_1| > i$, then consider x_k, y_k $(k \neq 1, 2)$. If there is some k satisfying $x_k \neq y_k$, then choose $s = (x_1, x_2, \ldots, y_k, \ldots, x_d), t = (y_1, y_2, \ldots, x_k, \ldots, y_d)$. If $x_k = y_k = p$ for all k, then put $s = (x_1, x_2, q, \ldots, q)$,
$t = (y_1, y_2, q, \ldots, q)$, where $q \neq p$ and $|q| \leq n$.

In all cases, $\{x, y, s, t\} \subset Q$ is a rectangular quadruple; so, x, y enjoy the rq-property in Q. $\square$

In $\mathbb{R}^3$, $Q(n) \setminus Q(i)$ $(0 \leq i \leq n-1)$ determines the sets $\cup\{ab \subset \mathbb{R}^3 : \|a-b\| = 1, a, b \in Q(n) \setminus Q(i)\}$ and $\cup\{abcd \subset \mathbb{R}^3 : abcd$ is a unit square, $a, b, c, d \in Q(n) \setminus Q(i)\}$; these sets are not rq-convex but rq-starshaped. The points $(\pm n, \pm n, \pm n)$ are in the kernel.

Starting with an abstract finite graph G, with $V(G)$ and $E(G)$ as vertex- and edge-set, respectively, we take $V(G)$ to be a set in $\mathbb{R}^2$, and each edge a line-segment joining its incident vertices, such that any two such line-segments meet in at most one point which is a vertex for both. So we obtain the *geometric graph* $G_1 = \cup\{e : e \in E(G)\} \subset \mathbb{R}^2$. Thus, a geometric graph in $\mathbb{R}^2$ is a finite union of line-segments. Edges do not cross. We identify G with G_1 [9].

Let $\mathcal{L} \subset \mathbb{R}^2$ be the infinite square lattice graph. It has $\mathbb{Z}^2$ as vertex set and all pairs of $\mathbb{Z}^2 \times \mathbb{Z}^2$ determining line-segments of unit length as edge set. Take in $\mathcal{L}$ some finite cycle C, considered as a geometric graph, and consider the geometric graph, called *grid graph*, the vertices and edges of which are all vertices and edges lying on C or inside the bounded plane region of boundary C [9].

Let V_m (resp. H_n) in $\mathcal{L}$ be the lattice-point set containing the lattice points from the origin to $(0, m)$ (resp. $(n, 0)$) on the $y-$axis (resp. $x-$axis) and V_{mn} the Cartesian product of V_m, H_n.

A grid graph is called a *rectangular grid graph*, if its vertex set is isometric to V_{mn} for some $m, n \geq 1$.

Obviously, the vertex set of any rectangular grid graph is rq-convex. Are there any other grid graphs with rq-convex vertex sets? Let G be a grid graph. For $\text{card}(V(G)) = 12$, there exists a further example, see Figure 1.

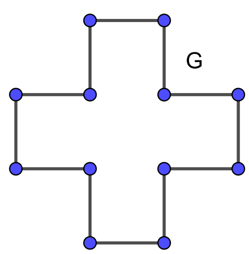


Figure 1: $V(G)$ is rq-convex.

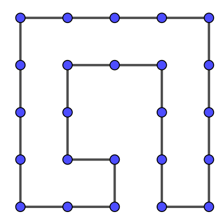


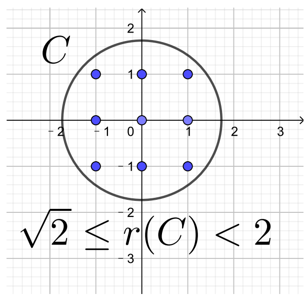
Figure 2: $Q(2) \setminus \{\mathbf{0}\}$

If $n \geq 2$, then $Q(n) \setminus \{\mathbf{0}\}$ is the vertex set of a grid graph. Therefore, $Q(n) \setminus \{\mathbf{0}\}$ is another example, and $\text{card}(Q(n) \setminus \{\mathbf{0}\}) = 4n(n+1) \geq 24$.

Conjecture 1. *The vertex set different from* $Q(n)\setminus\{\mathbf{0}\}$ *of a grid graph* G *with* $\text{card}(V(G)) > 12$ *is rq-convex, if and only if* G *is a rectangular grid graph.*

Now, we want to see what happens inside of discs considered in the Euclidean norm.

Let $C \subset \mathbb{R}^2$ be a circle with radius $r(C)$ and centre $\mathbf{0}$, and $V_C \subset \mathbb{Z}^2$ the set of all lattice points in $\text{conv}C$. This set V_C is the vertex set of a rectangular grid graph, if and only if $r(C) \in [\sqrt{2}, 2) \cup [2\sqrt{2}, \sqrt{2}+2)$. See Figure 3.



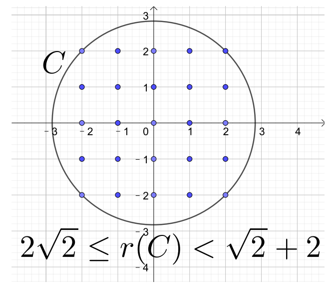


Figure 3: V_C is a rectangular grid graph

Theorem 2. *If $C \subset \mathbb{R}^2$ is a circle with centre* $\mathbf{0}$ *and radius at least $\sqrt{2}$, then V_C is rq-starshaped.*

Proof. Let a_x, a_y be the coordinates of $a \in V_C$. Thus, $a = (a_x, a_y)$. Consider the point $s = (s_x, 0) \in V_C$ with maximal s_x.

Case 1. $(s_x, 1) \in V_C$.

We prove that $\mathbf{0}$ belongs to the kernel of V_C. Indeed, for any point $c = (c_x, c_y) \in V_C$ with $c_x \neq 0$ and $c_y \neq 0$, the points $c, (c_x, 0), \mathbf{0}, (0, c_y)$ form a rectangular quadruple.

For any point $(c_x, 0) \in V_C$ with $c_x \neq 0$, the points $(c_x, 0), (c_x, 1), (0, 1), \mathbf{0}$ form a rectangular quadruple. The case of $(0, c_y) \in V_C$ with $c_y \neq 0$ is analogous.

Case 2. $(s_x, 1) \notin V_C$ and s_x is even.

We again prove that $\mathbf{0}$ belongs to the kernel of V_C. For any point $c = (c_x, c_y) \in V_C$ with $| c_x |< s_x$, the argument is the same as in Case 1.

For $| c_x |= s_x$, say $c_x = s_x$, we have the rectangular quadruple $\{(s_x, 0), (s_x/2, s_x/2), \mathbf{0}, (s_x/2, -s_x/2)\}$.

Case 3. $(s_x, 1) \notin V_C$ and s_x is odd.

We now prove that $(1, 0)$ belongs to the kernel of V_C. For any point $c = (c_x, c_y) \in V_C$ with $| c_x |< s_x$ and $| c_y |< s_x$, the argument is very similar to the one in Case 1.

For $c_x = s_x$ and $c_y = 0$, we have the rectangular quadruple $\{(s_x, 0), ((s_x + 1)/2, (s_x + 1)/2), (1, 0), ((s_x + 1)/2, -(s_x + 1)/2)\}$.

For $c_x = -s_x$ and $c_y = 0$, we have the rectangular quadruple $\{(-s_x, 0), ((-s_x + 1)/2, (-s_x + 1)/2), (1, 0), ((-s_x + 1)/2, (s_x - 1)/2)\}$.

For $c_x = 0$ and $c_y = s_x$, a suitable rectangular quadruple is $\{(0, s_x), ((-s_x + 1)/2, (s_x + 1)/2), (1, 0), ((s_x + 1)/2, (s_x - 1)/2)\}$.

For $c_x = 0$ and $c_y = -s_x$, we exhibit the rectangular quadruple $\{(0, -s_x), ((-s_x + 1)/2, -(s_x + 1)/2), (1, 0), ((s_x + 1)/2, (-s_x + 1)/2)\}$.

Hence, V_C is rq-starshaped. □

If the radius of C is smaller than $\sqrt{2}$, then there is no rectangular quadruple containing $\mathbf{0}$.

For a grid graph $G \subset \mathcal{L}$, let $G_2 \subset \mathbb{R}^2$ be the union of all unit squares, all edges of which are in G_1.

A set $S \subset \mathbb{R}^2$ is *horizontally convex* (*vertically convex*), if S includes every horizontal (vertical) line-segment with endpoints in S.[4]

Theorem 3. *Let $G \subset \mathcal{L}$ be a grid graph. If G_2 is horizontally convex and symmetric with respect to a vertical line containing lattice points, then $V(G)$ and G_1 are rq-starshaped.*

Proof. Suppose that the origin $\mathbf{0}$ is in the vertical axis of symmetry of G_2, i.e. G_2 is symmetric with respect to the y-axis Y. There exist at least two points $w, w' \in V(G)$ with the largest x-coordinate. Then choose points $k, k' \in Y$ such that k, w have the same y-coordinate, and k', w' have the same y-coordinate, too. Now, we show that k is in the kernel of both $V(G)$ and G_1.

For any point $v \in G_1 \setminus \overline{wk}$, let $v' \in \overline{kw}$ satisfy $vv' \perp kw$ and $v'' \in Y$ satisfy $vv'' \perp Y$. If $v \in G_1 \cap \overline{wk}$, take $v' \in \overline{k'w'}$ such that $vv' \perp kw$ and $v'' \in Y$ such that $v'v'' \perp Y$. Then, $\{v, v', k, v''\}$ is a rectangular quadruple in G_1; thus G_1 is rq-starshaped.

If, in particular, $v \in V(G)$, then $\{v, v', k, v''\} \subset V(G)$; so $V(G)$ is rq-starshaped, too. □

4 rq-Convexity of cubic lattice graphs

We say that a surface $S \subset \mathbb{R}^3$ is a *Jordan surface*, if S is the image of an injective continuous map of the sphere (boundary of a ball) into $\mathbb{R}^3$.

Consider the infinite cubic lattice $\mathbb{Z}^3 \subset \mathbb{R}^3$. Let the Jordan surface S be a finite union of unit squares with vertices in $\mathbb{Z}^3$. Consider the bounded component D of $\mathbb{R}^3 \setminus S$. The 2-complex S^* of all vertices of $\mathbb{Z}^3$, unit edges and unit squares with vertices in $\mathbb{Z}^3$ lying in cl$D$ is called a *grid* 2*-complex*. The union S_2 of all squares of S^* is a *geometric grid* 2*-complex*. The union S_1 of all edges of S^* is a *geometric grid* 1*-complex*. The complex S^* has $S_0 = \mathbb{Z}^3 \cap \mathrm{cl}D$ as vertex set.

The Jordan surface $B_{rst} = \mathrm{bd}(\mathbf{0}(r,0,0) \times \mathbf{0}(0,s,0) \times \mathbf{0}(0,0,t))$ determines a geometric grid 1-complex $(B_{rst})_1 = E_{rst}$, and a geometric grid 2-complex $(B_{rst})_2 = S_{rst}$.

Theorem 4. *A geometric grid* 1*-complex is rq-convex if and only if it is isometric to E_{rst} for some r, s, t.*

Proof. Note that the 1-skeleton of any right parallelotope is rq-convex (Theorem 4.1 in [3]). Any two points in E_{rst} are lying on the 1-skeleton included in E_{rst} of some right parallelotope. The "if" part is settled.

Now we show the other implication. Let S be a Jordan surface as above. Suppose it is translated such that $S_1 \subset E_{rst}$ where r, s, t are smallest possible. We show that, if $S_1 \neq E_{rst}$, then S_1 is not rq-convex.

Assume that $S_1 \neq E_{rst}$. We claim that there exists a *z-path* which is defined as $e_1 \cup e_2 \cup e_3 \subset S_1$ with edges e_1, e_2, e_3 pairwise orthogonal, allowing another edge $e'_1 \not\subset S_1$ in the boundary of the square determined by e_1, e_2, orthogonal to e_2.

Let $S_2^+ = \{\cup abcd : abcd$ is a unit square, $ab, bc, cd, da \subset S_1\}$. Consider the unit cube C with vertices in $\mathbb{Z}^3$. Let $F_1, F_2, F_3, F'_1, F'_2, F'_3$ be the six facets of $C = abcdd'a'b'c'$, where

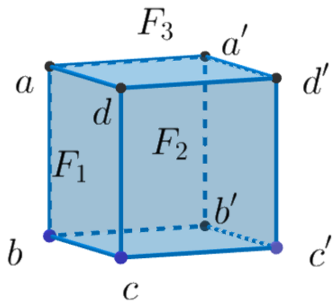


Figure 4: A cube C

$abcd \parallel a'b'c'd', aa' \parallel bb' \parallel cc' \parallel dd'$, and $F_1 = abcd, F_1' = a'b'c'd', F_2 = cdd'c', F_2' = abb'a', F_3 = add'a', F_3' = bcc'b'$. Remark that $S_2^+ = S_{rst}$ implies $S_1 = E_{rst}$.

To prove the claim we shall show that there exists a cube C in one of the following situations.

Case 1. Exactly two squares of C are not in S_2^+. In this case, the two squares are orthogonal, say F_1, F_2. Then only its edge cd is not in S_1. In this case we find the z-path $ab \cup bc \cup cc'$.

Case 2. Exactly three squares of C are not in S_2^+. If $F_1, F_2, F_3 \not\subset S_2^+$, then at least one of cd, ad, dd' say cd is not in $C \cap S_1$, and all but these three edges of C are in S_1. We find the z-path $ab \cup bc \cup cc'$. If $F_1, F_2, F_1' \not\subset S_2^+$, then $cd, c'd'$ are not in $C \cap S_1$. Then we find the z-path $ab \cup bc \cup cc'$.

Case 3. Exactly two non-opposite squares of C are in S_2^+. Let $F_1, F_2 \subset S_2^+$. Then at least one of $bb', b'c'$ is not in S_1, say $b'c' \not\subset S_1$; $F_1 \cup F_2$ contains the z-path $bc \cup cc' \cup c'd'$.

The set S_2^+ determines a set $\mathcal{C}$ of cubes. Let W be their union. Suppose W is not convex. By Tietze's theorem [6], W is not locally convex at some point p. Then p belongs to an edge of S_1, such that W is not locally convex at the endpoints u, u' of that edge. Clearly, u must be a vertex of at least 3 cubes $C_1, C_2, C_3 \in \mathcal{C}$, such that $uu' \subset C_1 \cap C_2 \cap C_3, uu' = C_1 \cap C_3$, but $uu' \subset \mathrm{bd}W$. Consider the 4-th cube C_4 containing uu'. Of course, $C_4 \not\subset W$. Let F_u be the facet of C_4 containing u, but not u', $F_{u'}$ the facet of C_4 containing u', but not u, and F the facet of C_4 not meeting C_1. If all these facets $F_u, F_{u'}, F \subset S_2^+$, then $C_4 \in \mathcal{C}$, absurd. So, either none of these facets lies in S_2^+ and C_4 is in Case 2 or Case 3, or precisely one of them lies in S_2^+, and C_4 is in Case 1 or Case 2, or precisely two of them lie in S_2^+, and C_4 is in Case 1.

Hence, if W is not convex, we are done. If W is convex, it is a parallelotope. The only parallelotope touching all sides of B_{rst} is $\mathrm{conv}B_{rst}$, and our claim is proven.

Hence, there is a z-path $e_1 \cup e_2 \cup e_3 \subset S$. Consider the edge $e_1' \parallel e_1$ of the square determined by e_1, e_2, and the edge $e_3' \parallel e_3$ of the square determined by e_2, e_3.

Let $\{v\} = e_2 \cap e_1, \{w\} = e_2 \cap e_3$. Take $x \in e_1, y \in e_3$ with $\|x - v\| \neq \|y - w\|$. Put $x' \in e_1'$ such that $xx' \perp e_1$ and $y' \in e_3'$ such that $yy' \perp e_3$. We have $\{x, y, x', y', v, w\} = C_{xy} \cap E_{rst}$. Hence, x, y don't have the rq-property in $C_{xy} \cap S_1$.

It is easily seen that, for any rectangle xyy^*x^* with $x^*, y^* \in S_1$, the points x^*, y^* cannot belong to unit cubes of the lattice near $e_1 \cup e_2 \cup e_3$. Rectangles vww^*v^* with $v^*, w^* \in S_1$

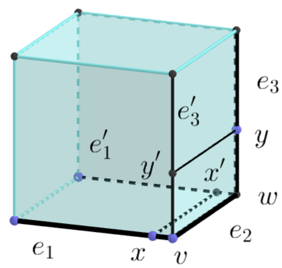


Figure 5: A z-path $e_1 \cup e_2 \cup e_3$

abound. But rectangles vyy^*v^* are already fewer, forming a finite family of rectangles parallel to e_1.

Now, taking x instead of v, but close to v, $(H_{xy} \cup H_{yx}) \cap S_1$ does not include rectangular quadruples any more, although it contains quadruples tending to rectangular ones as $x \to v$.

Hence, the presence of z-paths yields that S_1 is not rq-convex, and the theorem is proven. □

The statement concerning geometric grid 2-complexes analogous to Theorem 4 is false. Indeed, not only S_{rst} is rq-convex. For example, S_{999} minus the interior of the unit square in the middle of $\mathbf{0}(9,0,0) \times \mathbf{0}(0,9,0)$ is rq-convex, too.

The 0-dimensional analogon is also false. Let $R_0, R_1, R_2, R_3, R_4, R_5, R_6$ be seven right parallelotopes whose boundaries are in the union of all unit squares with vertices in $\mathbb{Z}^3$, such that, for every $i \in \{1, 2, \ldots, 6\}$, $R_0 \cup R_i$ is a right parallelotope. Then, $S = \text{bd}(\bigcup_{j=0}^{6} R_j)$ is also a Jordan surface. The set S_0 of all lattice points of S^* is rq-convex. Indeed, for any two points in S_0, there exists a plane parallel to some coordinate plane, such that the points symmetric about this plane are also in S_0.

5 rq-Convexity in triangular lattice graphs

Consider the Archimedean tiling (3^6) in $\mathbb{R}^2$, which is an infinite triangular lattice graph realised in the plane. We assume that its edges have length 1.

For a union D of triangles with boundaries in (3^6), let $W(D)$ be the set of all lattice points in D.

Let $I \subset \mathbb{R}^2$ be a line containing two adjacent lattice points in (3^6). We call I a *lattice line* of (3^6).

Theorem 5. *If I is a lattice line of (3^6), then the set of lattice points of (3^6) in a component of $\mathbb{R}^2 \setminus I$ is rq-convex.*

Proof. Let R be a component of $\mathbb{R}^2 \setminus I$. We take a Cartesian coordinate system as follows. The origin $\mathbf{0}$ should be a vertex of (3^6), and the x-axis a lattice line parallel to I, considered without loss of generality above it. Any lattice point in R has coordinates (x, y) with $x = m/2$ and $y = n\sqrt{3}/2 \geq 0$ and $m, n \in \mathbb{Z}$. Consider the point (x', y') with $x' = -3n$ and $y' = m\sqrt{3}$ if $x \geq 0$, and $x' = 3n$ and $y' = -m\sqrt{3}$ if $x < 0$. This is a lattice point in R. Moreover, $xx' + yy' = 0$, which shows that $\angle(x, y)\mathbf{0}(x', y') = \pi/2$. Any pair of vertices of (3^6) in R can be brought in the positions $\mathbf{0}, (x, y)$, by suitably choosing the Cartesian coordinate system. Hence, we find the right quadruple $\{(x, y), \mathbf{0}, (x', y'), (x + x', y + y')\}$ in R. □

Theorem 6. *If P is a regular hexagon of edge-length more than 1, with $\mathrm{bd}P \subset (3^6)$, then $W(P)$ is rq-starshaped.*

Proof. Suppose that the origin $\mathbf{0}$ is the centre of P and there is a side of P lying in horizontal direction. We say that the vertices in the same horizontal line are in the same floor. If the edge-length of P is $k \geq 2$, then there are $2k+1$ floors, where the first floor is in the bottom. Let F_i $(i = 1, 2, \dots, 2k+1)$ be the set of the vertices of $W(P)$ in the i-th floor. The set of the $k+i$ vertices in F_i and the corresponding $k+i$ vertices in F_{i+2} $(i = 1, 2, \dots, k)$ is rq-convex.

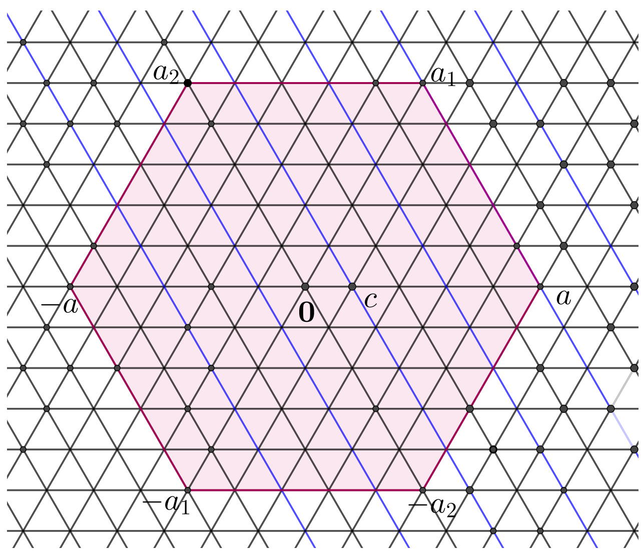


Figure 6: A regular hexagon with edge-length $k = 5$

If k is even, then $\mathbf{0}$ and any point in $(\bigcup_{i \text{ odd}} F_i) \setminus \{a, -a\}$ have the rq-property. We also consider the other two directions parallel to edges of P. Notice that there exists a direction such that a is in the first floor and $-a$ in the $(2k+1)$-th floor. For a suitable direction among the three as horizontal, every point of $W(P)$ is in an odd floor, and $\mathbf{0}$ is always in F_{k+1}. Hence, $\mathbf{0}$ is in the kernel of $W(P)$.

If k is odd, then choose $c \in F_{k+1}$ such that $\|c\| = 1$ and $\|c - a\| = k - 1$, as shown in Figure 6. In this case, there are two regular hexagons $P_1, P_2 \subset P$ such that c, a are opposite vertices of P_1 and $c, -a$ are opposite vertices of P_2. The edge-length of P_1 is $(k-1)/2$ and the edge-length of P_2 is $(k+1)/2$. Since the vertex set of any regular hexagon is rq-convex, c and a (resp. $-a$) have the rq-property. Hence, c and any point in $\bigcup_{i \text{ even}} F_i$ enjoy the rq-property. For a suitable direction among the other too, c and every point in $\bigcup_{j \text{ odd}} F_j$ are in odd floors. Hence, c is in the kernel of $W(P)$. □

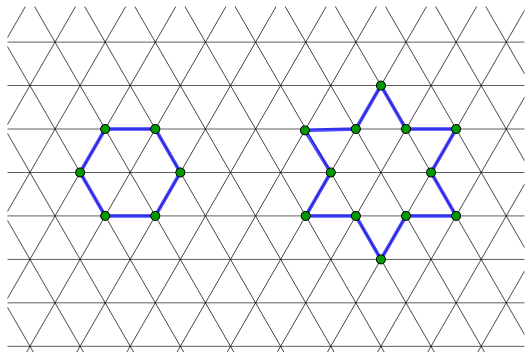


Figure 7: The vertex-sets of these cycles are rq-convex

Conjecture 2. *There are only two cycles in* (3^6) *the vertex-sets of which are rq-convex (see Figure 7).*

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