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**On the area and the lattice diameter of lattice triangles**

by

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**Abstract**

Given an integer $n \geq 2$, let $f(n)$ be the largest area a lattice triangle of lattice diameter at most $n$ may have. We prove that, if $n \geq 4$, then $f(n) \geq \frac{1}{2}(n^2+3)$, and $f(n) \geq \frac{19}{32}n^2 > \frac{1}{2}(n^2+3)$ for infinitely many $n$.

As a corollary, given any non-negative integer $N$, the largest possible area of a lattice triangle of lattice diameter $n$ is greater than $\frac{19}{32}(n+N)^2$ for infinitely many $n$.

**Key Words**: Lattice diameter, lattice triangles.
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# 1 Introduction

A *lattice point* is one in the Cartesian plane whose coordinates are both integral. A *convex lattice polygon* is the convex hull of at least three non-collinear lattice points. The *lattice diameter* of a convex lattice polygon is the maximal number of collinear lattice points contained in that polygon [3, 4].

A convex lattice polygon $P$ of lattice diameter $n$ contains at most $n^2$ lattice points [2, 3, 5]. By Pick's area formula, the area of $P$ is then at most $n^2 - \frac{5}{2}$. The area maxima are all known for $n$ in the range 2 through 5 [1, 3]. If $n \geq 6$, a maximal area $P$ has at least four vertices and its area is at most $n^2 - 3$ [1, 3]; moreover, the maximal area is at least $n^2 - 5$, and it has been conjectured that this would be the hoped for maximum [3].

Our purpose here is to deal with the largest possible area of a lattice triangle of lattice diameter *at most* $n$. Let $\mathcal{T}_n$ be the collection of all such triangles. For every triangle $T$ in $\mathcal{T}_n$, let $f(n,T)$ denote the area of $T$, and let $f(n) = \max_{T \in \mathcal{T}_n} f(n,T)$. By the preceding, $f(n) \leq n^2 - \frac{5}{2}$ or $f(n) \leq n^2 - 3$ if $n \geq 6$. To the best of our knowledge, these are the only known upper bounds and we failed to provide any better. Thus, we turned to lower bounds to prove that, if $n \geq 4$, then $f(n) \geq \frac{1}{2}(n^2+3)$, and $f(n) \geq \frac{19}{32}n^2 > \frac{1}{2}(n^2+3)$ for infinitely many $n$. (This latter shows that, if $f(n) \leq an^2 + bn + c$ for all sufficiently large integers $n$, then $a \geq \frac{19}{32} > \frac{1}{2}$.)

As a corollary, given any non-negative integer $N$, the largest possible area of a lattice triangle of lattice diameter $n$ is greater than $\frac{19}{32}(n+N)^2$ for infinitely many $n$.

# 2 The lower bounds

**Theorem.** *If $n \geq 4$, then $f(n) \geq \frac{1}{2}(n^2+3)$. Moreover, $f(n) \geq \frac{19}{32}n^2 > \frac{1}{2}(n^2+3)$ for infinitely many $n$.*

**Proof.** The proof of the first statement is part of the proof of the second, so we proceed to prove this latter. The idea is to consider a sequence of homothetic images of a suitable triangle and show that each lies in the desired collection.

Fix an integer $n \geq 4$. For every positive integer $k$, let $n_k = kn$, and let $T_k$ be the triangle with vertices at $O = (0, 0)$, $A_k = (k(n+1), 2k)$ and $B_k = (k(n-1), k(n+1))$. Clearly, $T_k$ is the factor $k$ homothetic image of $T_1$ from $O$. The area of $T_k$ is

$$\frac{1}{2}\left(1 + \frac{3}{n^2}\right) n_k^2.$$

We will show that $T_k$ is a member of $\mathcal{T}_{n_k}$, so $f(n_k) \geq f(n_k, T_k) = \frac{1}{2}\left(1 + \frac{3}{n^2}\right) n_k^2$. In particular, $f(n) = f(n_1) \geq f(n_1, T_1) = \frac{1}{2}(n_1^2 + 3) = \frac{1}{2}(n^2 + 3)$. This establishes the first statement.

For the second, notice that the coefficient of $n_k^2$ is maximised at $n = 4$, where it achieves the value $\frac{19}{32}$. In this case, $n_k = 4k$, and $f(n_k) \geq f(n_k, T_k) = \frac{19}{32} n_k^2 > \frac{1}{2}(n_k^2 + 3)$ for all $k \geq 2$, proving the second statement.

To show that $T_k$ is a member of $\mathcal{T}_{n_k}$, let $(a, b)$ and $(a', b')$ be distinct lattice points in $T_k$. Since the number of lattice points along the closed segment joining $(a, b)$ to $(a', b')$ is $\gcd(a - a', b - b') + 1$, we are to prove that $\gcd(a - a', b - b') < n_k = kn$.

To this end, we will show that, if one of the absolute values $|a - a'|$, $|b - b'|$ is greater than or equal to $kn$, then the other is positive and smaller than $kn$. Only the case $|a - a'| \geq kn$ will be considered; with minor computational changes, the case $|b - b'| \geq kn$ is dealt with similarly.

Let $|a - a'| \geq kn$. Then one of the points, say $(a, b)$, lies on one of the verticals $x = i$, $i = 0, 1, \ldots, k$, and the other, $(a', b')$, lies on one the verticals $x = kn + j$, $j = a, \ldots, k$. Notice that $a' \geq a(n + 1)$; equality holds here if and only if $a = k$ and $a' = k(n + 1)$.

The vertical $x = a$ crosses $OA_k$ and $OB_k$ at heights $2a/(n + 1)$ and $(n + 1)a/(n - 1) \geq 2a/(n + 1)$, respectively, so $2a/(n + 1) \leq b \leq a(n + 1)/(n - 1)$.

The vertical $x = a'$ crosses $OA_k$ and $A_kB_k$ at heights $2a'/(n + 1)$ and

$$\frac{1}{2}\big(k(n^2 + 3) - a'(n - 1)\big) \geq \frac{2a'}{n + 1},$$

respectively, so $b' \geq 2a'/(n + 1)$ and

$$b' \leq \frac{1}{2}\big(k(n^2 + 3) - a'(n - 1)\big) \leq \frac{1}{2}\big(k(n^2 + 3) - kn(n - 1)\big) = \frac{1}{2}k(n + 3) < kn,$$

on account of $a' \geq kn$ and $n \geq 4$. Hence $b' - b \leq b' < kn$.

Finally, recall that $a' \geq a(n + 1)$, to bound $b' - b$ from below:

$$b' - b \geq \frac{2a'}{n + 1} - \frac{(n + 1)a}{n - 1} \geq \frac{2a'}{n + 1} - \frac{a'}{n - 1} = \frac{a'(n - 3)}{n - 1} > 0,$$

on account of $a' \geq kn > 0$ and $n \geq 4$. This completes the argument.

Let $\mathcal{T}'_d$ be the subcollection of $\mathcal{T}_d$ consisting of all triangles of lattice diameter $d$, and let $f'(d) = \max_{T \in \mathcal{T}'_d} f(d, T)$.

**Corollary.** *Given any non-negative integer $N$, $f'(d) > \frac{19}{32}(d+N)^2$ for infinitely many $d$.*

**Proof.** In the above setting, each $T_k$ has exactly three lattice diameters: One along the horizontal through $A_k$, one along the vertical through $B_k$, and one along the first bisectrix. The vertical through $B_k$ crosses $OA_k$ at height $2k(n-1)/(n+1)$, so the lattice diameter $d_k$ of $T_k$ is

$$d_k = \left\lfloor k(n+1) - \frac{2}{n+1}\,k(n-1) \right\rfloor + 1 = \left(1 - \frac{1}{n}\right) n_k + \left\lfloor \frac{4n_k}{n(n+1)} \right\rfloor + 1.$$

It is then easily seen that

$$\frac{n^2+3}{n(n+1)}\,n_k \le d_k \le \frac{n^2+3}{n(n+1)}\,n_k + 1.$$

These inequalities show that $d_{k+1} - d_k > n - 2 \ge 2$, so the $d_k$ form a strictly increasing sequence of positive integers.

Given any non-negative integer $N$, the inequality on the right shows that $d_k < n_k - N$ for all but finitely many indices $k$, e. g., for all $k > 5(N+1)$.

Finally, set $n = 4$, to get $f'(d_k) \ge f(d_k, T_k) = f(n_k, T_k) = \frac{19}{32} n_k^2 > \frac{19}{32}(d_k + N)^2$ for all large enough $k$. The conclusion then follows by recalling that the $d_k$ form a strictly increasing sequence of positive integers.

**Remark.** If $n = 2$ or $3$, the pattern yields initial triangles of lattice diameter 3, respectively 4, and too small an area. The subsequent triangles provide weaker area lower bounds.

If $n = 2$, then $f(2) = \frac{3}{2} = 2^2 - \frac{5}{2} < \frac{1}{2}(2^2 + 3)$ is achieved by the triangle with vertices at $(0,0)$, $(1,0)$ and $(2,3)$.

If $n = 3$, the triangle with vertices at $(0,0)$, $(2,0)$ and $(3,4)$ has lattice diameter 3 (there are four such) and area $4 < \frac{1}{2}(3^2 + 3)$. Trying to maximise area over $\mathcal{T}_3$, we gathered quite strong evidence supporting the claim: $f(3) = f'(3) = 4$ and the maximal area triangles are all affine unimodular images of the one above.

We end with a word on convex lattice polygons. A *lattice triangulation* of such a polygon is one whose triangles are all lattice triangles (e. g., by non-crossing diagonals). Let $P$ be a lattice polygon of lattice diameter at most $n$. Then every lattice triangulation of $P$ contains at most one triangle of area at least $\frac{1}{2}(n^2+3)$ and hence at most one of area $f(n)$. Otherwise, the area of $P$ would be at least $n^2 + 3$, contradicting the area upper bound $n^2 - \frac{5}{2}$ mentioned in the introduction.

Finally, let $P_k$ be the parallelogram obtained from $T_k$ by reflecting this latter across the midpoint of one of its sides. The area of $P_k$ is $\left(1 + \frac{3}{n^2}\right) n_k^2$. On the other hand, letting $d_k$ denote the lattice diameter of $P_k$, the area of $P_k$ is at most $d_k^2 - \frac{5}{2}$. Consequently, if $0 < a < \left(1 + \frac{3}{n^2}\right)^{1/2}$, then the $d_k - an_k$ form an unbounded sequence.

## References

[1] E. Alarcon, Convex lattice polygons, Ph.D. Dissertation, University of Illinois at Urbana-Champaign (1994).

[2] E. Alarcon, An extremal result on convex lattice polygons, *Discr. Math.* **190** (1998), 227-234.

[3] I. Bárány, Z. Füredi, On the lattice diameter of a convex polygon, *Discr. Math.* **241** (2001), 41-50.

[4] C. E. Corzatt, Some extremal problems of number theory and geometry, Ph.D. Dissertation, University of Illinois at Urbana-Champaign (1974).

[5] S. Rabinowitz, A theorem about collinear lattice points, *Utilitas Math.* **36** (1986), 93-95.

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