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**Fubini theorem for conditional Fourier–Feynman transforms associated with random vectors**

by

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**Abstract**

In this paper, we use a vector-valued conditioning function to define a conditional Fourier–Feynman transform (CFFT) on the Wiener space. We establish the existence of the CFFT for bounded functionals which form a Banach algebra. We then investigate Fubini theorems for the CFFT. The Fubini theorems for the transforms investigated in this paper are to express the iterated CFFT as a single CFFT. The conditioning functions in the Fubini theorems are uncorrelated finite-dimensional random vectors on the Wiener space.

**Key Words**: Wiener space, conditional Fourier–Feynman transform, uncorrelated random vector, Fubini theorem.
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# 1 Introduction

The Feynman–Kac functionals are given by $F_t(x) = \exp\{\int_0^t \theta(s, x(s))ds\}$ where $\theta$ is a complex-valued potential on $[0,T]\times\mathbb{R}$. The conditional Feynman integrals of the Feynman–Kac functionals are important in a branch of the study of the Schrödinger equation. By a Feynman–Kac formula, many physical problems concerning the Schrödinger equation can be represented in terms of the conditional Feynman integral $E^{\mathrm{anf}_q}(F_t|X_t)$ of the Feynman–Kac functional $F_t$, where $X_t(x) = x(t)$. Moreover, the conditional Feynman integral provides solutions of the integral equations which are formally equivalent to the Schrödinger equation [10, 12, 18, 22, 23, 28]. We are obliged to point out that the conditional Feynman integral was defined in terms of the conditional Wiener integral. Based on this background, evaluation formulas for conditional Wiener integrals have been established through the papers [22, 23, 24, 28]. For a detailed survey of the conditional Wiener and Feynman integrals, see [8].

On the other hand, the Fourier–Feynman transform theory is very important in the study of infinite dimensional analysis. The theory of the analytic Fourier–Feynman transform suggested by Brue [1] now is playing a central role in the analytic Feynman integration theory and its applications. The classical Fourier–Feynman transform and several analogies have been improved in various research articles. For instance, see [2, 5, 6, 7, 13, 14, 15, 16]. This transform and its properties are similar in many respects to the ordinary Fourier transform of functions on Euclidean space. For an elementary introduction of the Fourier–Feynman transform, see [26].

Studies of conditional Wiener and Feynman integrals given finite dimensional conditioning functions were performed with additional topics in [9, 11, 12, 22, 28]. The concept of a CFFT was suggested by Park and Skoug in [25]. The structure of the CFFT is based on the conditional Wiener and Feynman integrals. In [25], using a one-dimensional conditioning function $X$ on the classical Wiener space $C_0[0,T]$, the space of continuous functions on the time interval $[0,T]$ such that $x(0)=0$, Park and Skoug defined the CFFT, $T_q(F|X)$, of functionals $F$ on $C_0[0,T]$. Since then, the theory of CFFTs for functionals on $C_0[0,T]$ has been developed by many authors. For example, see [6, 8, 9].

In view of these background illustrated above, it is worth-while to improve the study of conditional analytic Feynman integrals and CFFTs for functionals on the Wiener space $C_0[0,T]$. As a development of these integrals and transforms, we in this paper investigate some aspect of the conditional analytic Feynman integral and the CFFT for functionals on $C_0[0,T]$. The outline of this paper is as follows. In Section 3, we define the CFFT given a vector-valued conditioning function on $C_0[0,T]$. We then, in Section 4, provide explicit formulas for CFFTs of functionals in the Cameron and Storvick's Banach algebra $\mathcal{S}(L_2[0,T])$ [3]. Finally, in Section 5, we investigate some Fubini theorems involving the CFFTs (Theorems 5.1 and 5.2) and the conditional Feynman integral (Corollary 5.3 below). The conditioning functions in the Fubini theorems for the iterated CFFT are uncorrelated random vectors on the Wiener space $C_0[0,T]$.

## 2 Preliminaries

We now introduce basic concepts to define a CFFT for functionals on the complete Wiener measure space $(C_0[0,T],\mathcal{W}(C_0[0,T]),m_w)$, where $\mathcal{W}(C_0[0,T])$ denotes the $\sigma$-field of all Wiener measurable subsets. We denote the Wiener integral of a Wiener integrable functional $F$ by

$$E[F]\equiv E_x[F(x)]=\int_{C_0[0,T]}F(x)dm_w(x),$$

and for $u\in L_2[0,T]$ and $x\in C_0[0,T]$, we let $\langle u,x\rangle=\int_0^T u(t)dx(t)$ denote the Paley–Wiener–Zygmund (PWZ) stochastic integral [19, 20, 21]. It is well-known that for each $v\in L_2[0,T]$, the PWZ integral $\langle v,x\rangle$ exists for $m_w$-a.e. $x\in C_0[0,T]$ and is a Gaussian random variable with mean 0 and variance $\|v\|_2^2$. If $\{\alpha_1,\dots,\alpha_n\}$ is an orthogonal set of functions in $L_2[0,T]$, then the random variables, $\{\langle\alpha_j,x\rangle\}_{j=1}^n$, are independent.

Let $X$ be an $\mathbb{R}^n$-valued measurable function and let $Y$ be a $\mathbb{C}$-valued integrable functional on the complete Wiener space $(C_0[0,T],\mathcal{W}(C_0[0,T]),m_w)$. Let $\mathcal{F}(X)$ denote the $\sigma$-field generated by $X$. Then by the definition, the conditional expectation of $Y$ given $\mathcal{F}(X)$, written $E(Y|X)$, is any $\mathbb{C}$-valued $\mathcal{F}(X)-$measurable functional on $C_0[0,T]$ such that

$$\int_A Y(x)dm_w(x)=\int_A E(Y|X)(x)dm_w(x)\quad\text{for}\quad A\in\mathcal{F}(X).$$

It is well known that there exists a Borel measurable and $P_X-$integrable function $\psi$ on $(\mathbb{R}^n,\mathcal{B}(\mathbb{R}^n),P_X)$ such that $E(Y|X)=\psi\circ X$, where $\mathcal{B}(\mathbb{R}^n)$ denotes the Borel $\sigma$-field of Borel subsets in $\mathbb{R}^n$ and $P_X$ is the probability distribution of $X$ defined by $P_X(U)=m_w(X^{-1}(U))$ for $U\in\mathcal{B}(\mathbb{R}^n)$. The function $\psi(\vec{\xi})$, $\vec{\xi}\in\mathbb{R}^n$, is unique up to Borel null sets in $\mathbb{R}^n$. Following

Tucker [27] and Yeh [28], the function $\psi(\vec{\xi})$, written $E(Y|X=\vec{\xi})$, is called the conditional Wiener integral of $Y$ given $X$.

Let $\mathcal{N}=\{e_n\}_{n=1}^{\infty}$ be a countable orthonormal basis of $L_2[0,T]$. For each $n\in\mathbb{N}$, let $\gamma_n(x)=\langle e_n,x\rangle$ and let $\beta_n(t)=\int_0^t e_n(s)ds$ for $t\in[0,T]$. Then the PWZ stochastic integrals $\gamma_n(x)$, $n\in\mathbb{N}$, form a set of independent standard Gaussian random variables on $C_0[0,T]$ with $E_x[x(t)\gamma_n(x)]=\beta_n(t)$.

Let $\mathcal{G}=\{e_1^{\mathcal{G}},\dots,e_n^{\mathcal{G}}\}$ be a finite subset of $\mathcal{N}$. For each $e_j^{\mathcal{G}}\in\mathcal{G}$, $j\in\{1,\dots,n\}$, we denote $\langle e_j^{\mathcal{G}},x\rangle$ and $\int_0^t e_j^{\mathcal{G}}(s)ds$ by $\gamma_j^{\mathcal{G}}(x)$ and $\beta_j^{\mathcal{G}}(t)$, respectively. Given a finite subset $\mathcal{G}=\{e_1^{\mathcal{G}},\dots,e_n^{\mathcal{G}}\}$ of $\mathcal{N}$, let $\mathcal{H}_{\mathcal{G}}$ be the subspace of $L_2[0,T]$ spanned by $\mathcal{G}$, and let $X_{\mathcal{G}}:C_0[0,T]\longrightarrow\mathbb{R}^n$ be defined by

$$X_{\mathcal{G}}(x)=(\langle e_j^{\mathcal{G}},x\rangle,\dots,\langle e_n^{\mathcal{G}},x\rangle)=(\gamma_1^{\mathcal{G}}(x),\dots,\gamma_n^{\mathcal{G}}(x)). \tag{2.1}$$

Define a projection map $\mathcal{P}_{\mathcal{G}}$ from $L_2[0,T]$ into $\mathcal{H}_{\mathcal{G}}$ by

$$\mathcal{P}_{\mathcal{G}}v=\sum_{j=1}^{n}(v,e_j^{\mathcal{G}})_2e_j^{\mathcal{G}}\in\mathcal{H}_{\mathcal{G}}$$

where $(\cdot,\cdot)_2$ denotes the inner product on the Hilbert space $L_2[0,T]$. Then we see that the function $v-\mathcal{P}_{\mathcal{G}}v$ is in the orthogonal space $\mathcal{H}_{\mathcal{G}}^{\perp}$.

For each $x\in C_0[0,T]$ and $\vec{\xi}=(\xi_1,\dots,\xi_n)\in\mathbb{R}^n$, let

$$x_{\mathcal{G}}(t)=\langle\mathcal{P}_{\mathcal{G}}I_{[0,t]},x\rangle=\sum_{j=1}^{n}\gamma_j^{\mathcal{G}}(x)\beta_j^{\mathcal{G}}(t)$$

and

$$\vec{\xi}_{\mathcal{G}}(t)=\sum_{j=1}^{n}\xi_j(e_j^{\mathcal{G}},I_{[0,t]})_2=\sum_{j=1}^{n}\xi_j\beta_j^{\mathcal{G}}(t),$$

where $I_{[0,t]}$ denotes the indicator function of the interval $[0,t]$.

In [24], Park and Skoug proved the facts that the process $\{x(t)-x_{\mathcal{G}}(t),0\le t\le T\}$ and the Gaussian random variable $\gamma_j^{\mathcal{G}}(x)$ are stochastically independent for each $j\in\{1,\dots,n\}$, and that the processes $\{x(t)-x_{\mathcal{G}}(t),0\le t\le T\}$ and $\{x_{\mathcal{G}}(t),0\le t\le T\}$ are also stochastically independent. Using these basic results, Park and Skoug established the following evaluation formula to express conditional Wiener integrals in terms of ordinary Wiener integrals.

**Theorem 2.1** ([24])**.** *Let $F\in L_1(C_0[0,T])$. Then it follows that*

$$\begin{aligned}E(F|X_{\mathcal{G}}=\vec{\xi})&=E_x\big[F\big(x-x_{\mathcal{G}}+\vec{\xi}_{\mathcal{G}}\big)\big]\\&=E_x\bigg[F\bigg(x-\sum_{j=1}^{n}\gamma_j^{\mathcal{G}}(x)\beta_j^{\mathcal{G}}+\sum_{j=1}^{n}\xi_j\beta_j^{\mathcal{G}}\bigg)\bigg]\end{aligned} \tag{2.2}$$

*for a.e. $\vec{\xi}\in\mathbb{R}^n$.*

# 3 Conditional Fourier–Feynman transform

In order to define the CFFT, we need the concept of the scale-invariant measurability on the Wiener space. A subset $B$ of $C_0[0,T]$ is called a scale-invariant measurable (SIM) set if $\rho B \in \mathcal{W}(C_0[0,T])$ for all $\rho > 0$, and an SIM set $N$ is called a scale-invariant null set if $m_w(\rho N) = 0$ for all $\rho > 0$. A property which holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (SI-a.e.). A functional $F$ on $C_0[0,T]$ is said to be SIM provided $F$ is defined on an SIM set and $F(\rho\,\cdot)$ is $\mathcal{W}(C_0[0,T])$-measurable for every $\rho > 0$. For more detailed studies of the scale-invariant measurability, see [17].

The definition of the CFFT is based on the conditional analytic Wiener and the conditional Feynman integrals [11, 12, 25]. In this paper, we shall use exclusively the conditioning function $X_{\mathcal{G}}$ given by (2.1) to define a CFFT on $C_0[0,T]$.

Let $\mathbb{C}_+ = \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > 0\}$ and let $\widetilde{\mathbb{C}}_+ = \{\lambda \in \mathbb{C} \setminus \{0\} : \mathrm{Re}(\lambda) \geq 0\}$. Let $X_{\mathcal{G}} : C_0[0,T] \to \mathbb{R}^n$ be given by (2.1) and let $F$ be a $\mathbb{C}$-valued SIM functional such that the Wiener integral $E_x[F(\lambda^{-1/2}x)]$ exists as a finite number for all $\lambda > 0$. For $\lambda > 0$ and $\vec{\xi}$ in $\mathbb{R}^n$, let $J_F(\lambda;\vec{\xi}) = E(F(\lambda^{-1/2}\,\cdot)|X_{\mathcal{G}}(\lambda^{-1/2}\,\cdot) = \vec{\xi})$ denote the conditional Wiener integral of $F(\lambda^{-1/2}\,\cdot)$ given $X(\lambda^{-1/2}\,\cdot)$. If for a.e. $\vec{\xi} \in \mathbb{R}^n$, there exists a function $J_F^*(\lambda;\vec{\xi})$, analytic in $\mathbb{C}_+$ such that $J_F^*(\lambda;\vec{\xi}) = J_F(\lambda;\vec{\xi})$ for all $\lambda > 0$, then $J_F^*(\lambda;\,\cdot)$ is defined to be the conditional analytic Wiener integral of $F$ over $C_0[0,T]$ given $X_{\mathcal{G}}$ with parameter $\lambda$. For $\lambda \in \mathbb{C}_+$, we write

$$E^{\mathrm{anw}_\lambda}(F|X_{\mathcal{G}} = \vec{\xi}) = J_F^*(\lambda;\vec{\xi}).$$

If for fixed real $q \in \mathbb{R} \setminus \{0\}$, the limit

$$\lim_{\substack{\lambda \to -iq \\ \lambda \in \mathbb{C}_+}} E^{\mathrm{anw}_\lambda}(F|X_{\mathcal{G}} = \vec{\xi})$$

exists for a.e. $\vec{\xi} \in \mathbb{R}^n$, then we will denote the value of this limit by

$$E^{\mathrm{anf}_q}(F|X_{\mathcal{G}} = \vec{\xi}),$$

and we call it the conditional analytic Feynman integral of $F$ over $C_0[0,T]$ given $X_{\mathcal{G}}$ with parameter $q$.

Let $F$ be a $\mathbb{C}$-valued SIM functional on $C_0[0,T]$ such that the Wiener integral $E[F(y + \lambda^{-1/2}\,\cdot)] \equiv E_x[F(y + \lambda^{-1/2}x)]$ exists as a finite number for all $\lambda > 0$. Then one can easily see from (2.2) that for all $\lambda > 0$,

$$\begin{aligned} E(F(\lambda^{-1/2}\,\cdot)|X_{\mathcal{G}}(\lambda^{-1/2}\,\cdot) = \vec{\xi}) &\equiv E(F(\lambda^{-1/2}\,\cdot)|\gamma_j^{\mathcal{G}}(\lambda^{-1/2}\,\cdot) = \xi_j,\, j = 1,\dots,n) \\ &= E_x\left[F\left(\lambda^{-1/2}x - \lambda^{-1/2}\sum_{j=1}^n \gamma_j^{\mathcal{G}}(x)\beta_j^{\mathcal{G}} + \sum_{j=1}^n \xi_j\beta_j^{\mathcal{G}}\right)\right]. \end{aligned} \tag{3.1}$$

Thus we have that

$$E^{\mathrm{anw}_\lambda}(F|X_{\mathcal{G}} = \vec{\xi}) = E_x^{\mathrm{anw}_\lambda}\left[F\left(x - \sum_{j=1}^n \gamma_j^{\mathcal{G}}(x)\beta_j^{\mathcal{G}} + \sum_{j=1}^n \xi_j\beta_j^{\mathcal{G}}\right)\right],$$

and

$$E^{\mathrm{an}f_q}(F|X_n=\vec{\xi})=E_x^{\mathrm{an}f_q}\left[F\left(x-\sum_{j=1}^n\gamma_j^{\mathcal{G}}(x)\beta_j^{\mathcal{G}}+\sum_{j=1}^n\xi_j\beta_j^{\mathcal{G}}\right)\right],$$

where $E_x^{\mathrm{an}w_\lambda}[F(x)]$ and $E_x^{\mathrm{an}f_q}[F(x)]$ denote the analytic Wiener and the analytic Feynman integrals of functionals $F$ on $C_0[0,T]$, respectively, see [3, 12, 13, 14, 15, 16].

We are now ready to state the definitions of the CFFT of functionals on $C_0[0,T]$.

**Definition 3.1.** *Let $F:C_0[0,T]\to\mathbb{C}$ be an SIM functional on $C_0[0,T]$ such that the Wiener integral $E[F(y+\lambda^{-1/2}\cdot)]$ exists as a finite number for all $\lambda>0$. Let $X_{\mathcal{G}}:C_0[0,T]\to\mathbb{R}^n$ be given by (2.1). For $\lambda\in\mathbb{C}_+$ and $y\in C_0[0,T]$, let $T_\lambda(F|X_{\mathcal{G}})(y,\vec{\xi})$ denote the conditional analytic Wiener integral of $F(y+\cdot)$ given $X_{\mathcal{G}}$, that is to say,*

$$\begin{aligned}T_\lambda(F|X_{\mathcal{G}})(y,\vec{\xi})&=E^{\mathrm{an}w_\lambda}(F(y+\ \cdot)|X_{\mathcal{G}}=\vec{\xi})\\&=E_x^{\mathrm{an}w_\lambda}\left[F\left(y+x-\sum_{j=1}^n\gamma_j^{\mathcal{G}}(x)\beta_j^{\mathcal{G}}+\sum_{j=1}^n\xi_j\beta_j^{\mathcal{G}}\right)\right].\end{aligned}$$

*We define the $L_1$ analytic CFFT $T_q^{(1)}(F|X_{\mathcal{G}})(y,\vec{\xi})$ of $F$ given $X_{\mathcal{G}}$ by the formula*

$$T_q^{(1)}(F|X_{\mathcal{G}})(y,\vec{\xi})=\lim_{\substack{\lambda\to-iq\\ \lambda\in\mathbb{C}_+}}T_\lambda(F|X_{\mathcal{G}})(y,\vec{\xi}).$$

# 4 Conditional Fourier–Feynman transform for functionals in a Banach algebra

In this section, we will establish the existences of the CFFT for bounded functionals in the Cameron and Storvick's Banach algebra $\mathcal{S}(L_2[0,T])$.

The Banach algebra $\mathcal{S}(L_2[0,T])$ consists of functionals $F_\sigma$ on $C_0[0,T]$ having the form

$$F_\sigma(x)=\int_{L_2[0,T]}\exp\{i\langle u,x\rangle\}d\sigma(u) \tag{4.1}$$

for SI-a.e. $x\in C_0[0,T]$, where the associated measure $\sigma$ is an element of the Banach algebra $\mathcal{M}(L_2[0,T])$, the space of $\mathbb{C}$-valued countably additive (and hence finite) Borel measures on $L_2[0,T]$. More precisely, since we shall identify functionals which coincide SI-a.e. on $C_0[0,T]$, the space $\mathcal{S}(L_2[0,T])$ can be regarded as the space of all s-equivalence classes of functionals of the form (4.1). It was also shown in [3] that the correspondence $f\mapsto F$ is injective, carries convolution into pointwise multiplication and that $\mathcal{S}(L_2[0,T])$ is a Banach algebra with the norm

$$\|F_\sigma\|\equiv\|\sigma\|=\int_{L_2[0,T]}d|\sigma|(u). \tag{4.2}$$

In particular, it was shown in [4] that the Banach algebra $\mathcal{S}(L_2[0,T])$ contains many functionals of interest in Feynman integration theory. For more details, see [3, 4].

Given a complex measure $\sigma\in\mathcal{M}(L_2[0,T])$, let

$$\mathbb{S}(\sigma)=\{\tilde{\sigma}\in\mathcal{M}(L_2[0,T]):\|\sigma\|=\|\tilde{\sigma}\|\}. \tag{4.3}$$

Using the fact that the PWZ stochastic integral $\langle u,x\rangle$ of a function $u$ in $L_2[0,T]$ is a Gaussian random variable, as a functional of $x\in C_0[0,T]$, with mean zero and variance $\|u\|_2^2$, and the change of variable theorem, we have the following results.

Let $F_\sigma\in\mathcal{S}(L_2[0,T])$ given by (4.1). Then it was shown that for all $q\in\mathbb{R}\setminus\{0\}$,

$$E^{\mathrm{an}f_q}[F_\sigma]=\int_{L_2[0,T]}\exp\Big\{-\frac{i}{2q}\|u\|_2^2\Big\}d\sigma(u),$$

and

$$T_q^{(1)}(F_\sigma)(y)=\int_{L_2[0,T]}\exp\Big\{i\langle u,y\rangle-\frac{i}{2q}\|u\|_2^2\Big\}d\sigma(u)$$

for SI-a.e. $y\in C_0[0,T]$, where $T_q^{(1)}(F)$ denotes the analytic Fourier–Feynman transform for functionals $F$ on $C_0[0,T]$, see [14]. We also refer to the article [5, 6, 7] for more interesting results of the analytic Fourier–Feynman transforms.

**Lemma 4.1.** *For each $u\in L_2[0,T]$ and any $\rho>0$, it follows that*

$$E_x[\exp\{i\rho\langle u,x\rangle\}]=\exp\big\{-\rho^2\|u\|_2^2\big\}. \tag{4.4}$$

**Lemma 4.2.** *Let $\mathcal{G}=\{e_1^{\mathcal{G}},\dots,e_n^{\mathcal{G}}\}$ be a subset of the complete orthonormal set $\mathcal{N}$ in $L_2[0,T]$. Then for each $u\in L_2[0,T]$ and any $\rho>0$, it follows that*

$$E_x\bigg[\exp\bigg\{i\rho\Big\langle u,x-\sum_{j=1}^n\gamma_j^{\mathcal{G}}(x)\beta_j^{\mathcal{G}}\Big\rangle\bigg\}\bigg]=\exp\bigg\{-\frac{\rho^2}{2}\bigg[\|u\|^2-\sum_{j=1}^n(u,e_j^{\mathcal{G}})^2\bigg]\bigg\}. \tag{4.5}$$

*In particular, it follows that for any $q\in\mathbb{R}\setminus\{0\}$ and any $\rho>0$,*

$$E_x^{\mathrm{an}f_q}\bigg[\exp\bigg\{i\rho\Big\langle u,x-\sum_{j=1}^n\gamma_j^{\mathcal{G}}(x)\beta_j^{\mathcal{G}}\Big\rangle\bigg\}\bigg]=\exp\bigg\{-\frac{i\rho^2}{2q}\bigg[\|u\|^2-\sum_{j=1}^n(u,e_j^{\mathcal{G}})^2\bigg]\bigg\}. \tag{4.6}$$

*Proof.* Using the bilinearity of the PWZ stochastic integral $\langle\cdot,\cdot\rangle$ and equation (4.4) with $u$ replaced with $u-\sum_{j=1}^n(u,e_j^{\mathcal{G}})_2e_j^{\mathcal{G}}$, equation (4.5) follows immediately. Next, in view of the definition of the analytic Feynman integral [3, 13, 14, 15], one can verify equation (4.6). ☐

In our first theorem of this section, we establish the existences of the CFFT $T_q^{(1)}(F|X_{\mathcal{G}})$ of functionals in the Banach algebra $\mathcal{S}(L_2[0,T])$.

**Theorem 4.3.** *Let $F_\sigma\in\mathcal{S}(L_2[0,T])$ be given by equation (4.1), and given an orthonormal subset $\mathcal{G}=\{e_1^{\mathcal{G}},\dots,e_n^{\mathcal{G}}\}$ of $\mathcal{N}$, let $X_{\mathcal{G}}$ be given by equation (2.1). Then for a.e. $\vec{\xi}\in\mathbb{R}^n$, it follows that*

$$\begin{aligned}&T_q^{(1)}(F|X_{\mathcal{G}})(y,\vec{\xi})\\&=\int_{L_2[0,T]}\exp\bigg\{i\langle u,y\rangle-\frac{i}{2q}\bigg[\|u\|_2^2-\sum_{j=1}^n(u,e_j^{\mathcal{G}})_2^2\bigg]+i\sum_{j=1}^n\xi_j(u,e_j^{\mathcal{G}})_2\bigg\}d\sigma(u)\end{aligned} \tag{4.7}$$

*for all $q\in\mathbb{R}\setminus\{0\}$ and SI-a.e. $y\in C_0[0,T]$.*

*Proof.* Using (4.1), (3.1) with $F$ replaced with $F_\sigma(y+\cdot)$, the Fubini theorem, (4.5) with $w$ and $\rho$ replaced with $u$ and $\lambda^{-1/2}$, it follows that for $(\lambda,\vec{\xi})\in(0,+\infty)\times\mathbb{R}^n$,

$$\begin{aligned}
&J_{F_\sigma(y+\cdot)}(\lambda;\vec{\xi}) \equiv E\big(F_\sigma(y+\lambda^{-1/2}\,\cdot\,)\big|X_{\mathcal{G}}(\lambda^{-1/2}\,\cdot\,)=\vec{\xi}\big)\\
&=E_x\left[F\left(y+\lambda^{-1/2}x-\lambda^{-1/2}\sum_{j=1}^n\gamma_j^{\mathcal{G}}(x)\beta_j^{\mathcal{G}}+\sum_{j=1}^n\xi_j\beta_j^{\mathcal{G}}\right)\right]\\
&=\int_{L_2[0,T]}\exp\left\{i\langle u,y\rangle+i\left\langle u,\sum_{j=1}^n\xi_j\beta_j^{\mathcal{G}}\right\rangle\right\}E_x\left[\exp\left\{i\lambda^{-1/2}\left\langle u,x-\sum_{j=1}^n\gamma_j^{\mathcal{G}}(x)\beta_j^{\mathcal{G}}\right\rangle\right\}\right]d\sigma(u)\\
&=\int_{L_2[0,T]}\exp\left\{i\langle u,y\rangle+i\left\langle u,\sum_{j=1}^n\xi_j\beta_j^{\mathcal{G}}\right\rangle\right\}E_x\left[\exp\left\{i\lambda^{-1/2}\left\langle u-\sum_{j=1}^n(u,e_j)_2e_j,x\right\rangle\right\}\right]d\sigma(u)\\
&=\int_{L_2[0,T]}\exp\left\{i\langle u,y\rangle-\frac{1}{2\lambda}\left[\|u\|_2^2-\sum_{j=1}^n(u,e_j^{\mathcal{G}})_2^2\right]+i\sum_{j=1}^n\xi_j(u,e_j^{\mathcal{G}})_2\right\}d\sigma(u).
\end{aligned}$$

Let

$$J^*_{F_\sigma(y+\cdot)}(\lambda;\vec{\xi})=\int_{L_2[0,T]}\exp\left\{i\langle u,y\rangle-\frac{1}{2\lambda}\left[\|u\|_2^2-\sum_{j=1}^n(u,e_j^{\mathcal{G}})_2^2\right]+i\sum_{j=1}^n\xi_j(u,e_j^{\mathcal{G}})_2\right\}d\sigma(u) \tag{4.8}$$

for $\lambda\in\mathbb{C}_+$. Since $\mathrm{Re}(\lambda)>0$ for all $\lambda\in\mathbb{C}_+$, it follows that

$$\begin{aligned}
&\big|J^*_{F_\sigma(y+\cdot)}(\lambda;\vec{\xi})\big|\\
&\le\int_{L_2[0,T]}\left|\exp\left\{i\langle u,y\rangle-\frac{1}{2\lambda}\left[\|u\|_2^2-\sum_{j=1}^n(u,e_j^{\mathcal{G}})_2^2\right]+i\sum_{j=1}^n\xi_j(u,e_j)_2\right\}\right|d|\sigma|(u)\\
&\le\int_{L_2[0,T]}d|\sigma|(u)\\
&=\|\sigma\|<+\infty.
\end{aligned} \tag{4.9}$$

Hence, applying the dominated convergence theorem, we see that $J^*_{F_\sigma(y+\cdot)}(\lambda;\vec{\xi})$ is a continuous function of $\lambda\in\widetilde{\mathbb{C}}_+$. Since

$$K(\lambda)\equiv\exp\left\{i\langle u,y\rangle-\frac{1}{2\lambda}\left[\|u\|_2^2-\sum_{j=1}^n(u,e_j)_2^2\right]+i\sum_{j=1}^n\xi_j(u,e_j^{\mathcal{G}})_2\right\}$$

is analytic on $\mathbb{C}_+$, using the Fubini theorem, it follows that

$$\int_\Gamma J^*_{F_\sigma(y+\cdot)}(\lambda;\vec{\xi})d\lambda=\int_{L_2[0,T]}\int_\Gamma K(\lambda)d\lambda d\sigma(u)=0$$

for all rectifiable closed curves $\Gamma$ lying in $\mathbb{C}_+$. Thus, by the Morera theorem, we see that $J^*_{F_\sigma(y+\cdot)}(\lambda;\vec{\xi})$ is analytic in $\lambda\in\mathbb{C}_+$. Therefore, the conditional analytic Wiener integral

$$T_\lambda(F_\sigma|X_{\mathcal{G}})(y,\vec{\xi})=E^{\mathrm{anw}_\lambda}(F_\sigma(y+\cdot)|X_{\mathcal{G}}=\vec{\xi})=J^*_{F_\sigma(y+\cdot)}(\lambda;\vec{\xi})$$

exists and is given by the right-hand side of (4.8). Finally, by the dominated convergence theorem (the use of which is justified by (4.9)), the $L_1$ analytic CFFT $T_q^{(1)}(F_\sigma|X_{\mathcal{G}}=\vec{\xi})$ of $F_\sigma$ exists and is given by the formula (4.7). ∎

From the definition of the conditional analytic Feynman integral and the $L_1$ analytic CFFT, it follows that

$$T_q^{(1)}(F_\sigma|X_{\mathcal{G}})(0,\vec{\xi}) = E^{\mathrm{an}f_q}(F_\sigma|X_{\mathcal{G}}=\vec{\xi}). \tag{4.10}$$

We thus have the following corollary.

**Corollary 4.4.** *Let $F_\sigma$ and $X_{\mathcal{G}}$ be as in Theorem 4.3. Then the conditional analytic Feynman integral $E^{\mathrm{an}f_q}(F_\sigma|X_{\mathcal{G}}=\vec{\xi})$ of $F_\sigma$ exists for all $q\in\mathbb{R}\setminus\{0\}$ and a.e. $\vec{\xi}\in\mathbb{R}^n$, and is given by the formula*

$$E^{\mathrm{an}f_q}(F_\sigma|X_{\mathcal{G}}=\vec{\xi}) = \int_{L_2[0,T]} \exp\bigg\{-\frac{i}{2q}\bigg[\|u\|_2^2-\sum_{j=1}^n (u,e_j^{\mathcal{G}})_2^2\bigg]+i\sum_{j=1}^n \xi_j(u,e_j^{\mathcal{G}})_2\bigg\}d\sigma(u).$$

**Remark 4.5.** *Given a functional $F_\sigma$ in $\mathcal{S}(L_2[0,T])$ with the corresponding measure $\sigma\in\mathcal{M}(L_2[0,T])$, and given a nonzero real number $q$ and a vector $\vec{\xi}\in\mathbb{R}^n$, define a set function $\sigma_{q,\vec{\xi}}:\mathcal{B}(L_2[0,T])\to\mathbb{C}$ by the formula*

$$\sigma_{q,\vec{\xi}}(U) = \int_U \exp\bigg\{-\frac{i}{2q}\bigg[\|u\|_2^2-\sum_{j=1}^n (u,e_j^{\mathcal{G}})_2^2\bigg]+i\sum_{j=1}^n \xi_j(u,e_j^{\mathcal{G}})_2\bigg\}d\sigma(u) \tag{4.11}$$

*for each $U$ in $\mathcal{B}(L_2[0,T])$, the Borel $\sigma$-field on $L_2[0,T]$. Then $\sigma_{q,\vec{\xi}}$ is obviously a complex measure in $\mathcal{M}(L_2[0,T])$. One can easily see that the complex measure $\sigma_{q,\vec{\xi}}$ defined by (4.11) is an element of the sphere $\mathbb{S}(\sigma)$ in $\mathcal{M}(L_2[0,T])$ for any $q\in\mathbb{R}\setminus\{0\}$ and $\vec{\xi}\in\mathbb{R}^n$. Then equation (4.7) can be rewritten by*

$$T_q^{(1)}(F_\sigma|X_{\mathcal{G}})(y,\vec{\xi}) = \int_{L_2[0,T]} \exp\{i\langle u,y\rangle\}d\sigma_{q,\vec{\xi}}(u)$$

*for SI-a.e. $y\in C_0[0,T]$, and so the $L_1$ analytic CFFT $T_q^{(1)}(F_\sigma|X_{\mathcal{G}})(\,\cdot\,,\vec{\xi})$ of $F_\sigma$ with parameter $q$ is an element of $\mathcal{S}(L_2[0,T])$ for each $\vec{\xi}\in\mathbb{R}^n$.*

In view of Theorem 4.3 and Remark 4.5, we easily obtain the following theorem.

**Theorem 4.6.** *Let $F_\sigma$ and $X_{\mathcal{G}}$ be as in Theorem 4.3. Then,*

*(i) for any $q$ in $\mathbb{R}\setminus\{0\}$, it follows that*

$$T_{-q}^{(1)}(T_q^{(1)}(F_\sigma|X_{\mathcal{G}})(\,\cdot\,,\vec{\xi})|X_{\mathcal{G}})(y,-\vec{\xi}) = F(y)$$

*for SI-a.e. $y\in C_0[0,T]$ and a.e. $\vec{\xi}\in\mathbb{R}^n$; and* *(ii) for any finite sequence* $\{q_1,\dots,q_m\}$ *in* $\mathbb{R}\setminus\{0\}$ *which satisfies the condition*

$$\frac{1}{q_1}+\cdots+\frac{1}{q_k}\neq 0 \ \text{ for each } k\in\{1,\dots,m\}, \tag{4.12}$$

*it follows that*

$$\begin{aligned} &T^{(1)}_{q_m}\Big(T^{(1)}_{q_{m-1}}\Big(\cdots T^{(1)}_{q_1}(F_\sigma|X_{\mathcal{G}})(\,\cdot\,,\vec{\xi}^{(1)})\cdots\Big|X_{\mathcal{G}}\Big)(\,\cdot\,,\vec{\xi}^{(m-1)})\Big|X_{\mathcal{G}}\Big)(y,\vec{\xi}^{(m)})\\ &=T^{(1)}_{\alpha_m}(F_\sigma|X_{\mathcal{G}})\bigg(y,\sum_{k=1}^{m}\vec{\xi}^{(k)}\bigg) \end{aligned} \tag{4.13}$$

*for SI-a.e.* $y\in C_0[0,T]$ *and a.e.* $(\vec{\xi}^{(1)},\dots,\vec{\xi}^{(m)})$ *in* $(\mathbb{R}^n)^m$*, the product of* $m$ *copies of* $\mathbb{R}^n$*, where*

$$\alpha_m=\bigg(\frac{1}{q_1}+\cdots+\frac{1}{q_m}\bigg)^{-1}. \tag{4.14}$$

*Also, both of the expressions in* (4.13) *are given by the expression*

$$\int_{L_2[0,T]}\exp\bigg\{i\langle u,y\rangle-\frac{i}{2\alpha_m}\bigg[\|u\|_2^2-\sum_{j=1}^{n}(u,e_j^{\mathcal{G}})_2^2\bigg]+i\sum_{k=1}^{m}\sum_{j=1}^{n}\xi_j^{(k)}(u,e_j^{\mathcal{G}})_2\bigg\}d\sigma(u)$$

*for SI-a.e.* $y\in C_0[0,T]$.

Let $F$ be an SIM functional on $C_0[0,T]$. Define a transform $A_n:(C_0[0,T])^m\to C_0[0,T]$ by $A_n(x_1,\dots,x_n)=\sum_{j=1}^m x_j$. Then for $(x_1,\dots,x_m)\in(C_0[0,T])^m$,

$$F\bigg(\sum_{j=1}^{m}x_j\bigg)=F\circ A_m(x_1,\dots,x_m)$$

For each $j\in\{1,\dots,m\}$, the $x_j$-section of $F\circ A_m$ is SIM on $C_0[0,T]$, because $A_m$ is continuous on $(C_0[0,T])^m$. Next, for notational conveniences, we write the conditional analytic Feynman integral

$$E^{\mathrm{anf}_q}(F|X_{\mathcal{G}}=\vec{\eta})$$

by

$$E^{\mathrm{anf}_q}(F(x)|X_{\mathcal{G}}(x)=\vec{\eta})$$

as used in [22, 28].

Using these conventions and applying equation (4.10), we have the following Fubini theorem for the iterated conditional analytic Feynman integral.

**Corollary 4.7.** *Let* $F_\sigma$ *and* $X_{\mathcal{G}}$ *be as in Theorem 4.3. Then for any finite sequence* $\{q_1,\dots,q_m\}$ *in* $\mathbb{R}\setminus\{0\}$ *which satisfies the condition* (4.12)*, it follows that*

$$\begin{aligned} &E^{\mathrm{anf}_{q_m}}\Big(E^{\mathrm{anf}_{q_{m-1}}}\Big(\cdots\Big(E^{\mathrm{anf}_{q_1}}(F_\sigma\circ A_m(x_1,\dots,x_m)\Big|X_{\mathcal{G}}(x_1)=\vec{\xi}^{(1)}\Big)\\ &\qquad\qquad\cdots\Big|X_{\mathcal{G}}(x_{m-1})=\vec{\xi}^{(m-1)}\Big)\Big|X_{\mathcal{G}}(x_m)=\vec{\xi}^{(m)}\Big)\\ &=E^{\mathrm{anf}_{\alpha_m}}\bigg(F_\sigma(x)\bigg|X_{\mathcal{G}}(x)=\sum_{k=1}^{m}\vec{\xi}^{(k)}\bigg) \end{aligned}$$

*for a.e.* $(\vec{\xi}^{(1)},\dots,\vec{\xi}^{(m)})$ *in* $(\mathbb{R}^n)^m$*, where* $\alpha_m$ *is given by* (4.14).

# 5 Iterated conditional Fourier–Feynman transform: Fubini theorems

In this section, we establish Fubini theorems for the iterated CFFT. The conditioning functions in our Fubini theorems for the iterated CFFT are uncorrelated finite-dimensional random vectors on the Wiener space.

Given a finite subset $\mathcal{G}$ of $\mathcal{N}$, let $\{\mathcal{G}_k\}_{k=1}^m$ be a partition of $\mathcal{G}$. For each $k \in \{1,\dots,m\}$, say $\mathcal{G}_k = \{e_1^{\mathcal{G}_k},\dots,e_{n_k}^{\mathcal{G}_k}\}$. Let $X_{\mathcal{G}}$ be a conditioning function given by (2.1). Then we can rewrite $X_{\mathcal{G}}$ by

$$\begin{aligned} X_{\mathcal{G}}(x) &= X_{\mathcal{G}_1}(x) \wedge X_{\mathcal{G}_2}(x) \wedge \cdots \wedge X_{\mathcal{G}_m}(x) \\ &= (\langle e_1^{\mathcal{G}_1},x\rangle,\dots,\langle e_{n_1}^{\mathcal{G}_1},x\rangle) \wedge (\langle e_1^{\mathcal{G}_2},x\rangle,\dots,\langle e_{n_2}^{\mathcal{G}_2},x\rangle) \wedge \cdots \wedge (\langle e_1^{\mathcal{G}_m},x\rangle,\dots,\langle e_{n_m}^{\mathcal{G}_m},x\rangle) \\ &= (\langle e_1^{\mathcal{G}_1},x\rangle,\dots,\langle e_{n_1}^{\mathcal{G}_1},x\rangle, \langle e_1^{\mathcal{G}_2},x\rangle,\dots,\langle e_{n_2}^{\mathcal{G}_2},x\rangle,\dots,\langle e_1^{\mathcal{G}_m},x\rangle,\dots,\langle e_{n_m}^{\mathcal{G}_m},x\rangle). \end{aligned} \tag{5.1}$$

Let $\sigma$ be a complex measure in $\mathcal{M}(L_2[0,T])$. For a finite sequence $Q = \{q_1,\dots,q_m\}$ of nonzero real numbers which satisfies the condition (4.12), and a conditioning function $X_{\mathcal{G}}$ given by (5.1), define a complex measure $\sigma_{\mathcal{G}_1\cup\cdots\cup\mathcal{G}_m}^{(q_1,\dots,q_m)} : \mathcal{B}(L_2[0,T]) \to \mathbb{C}$ by

$$\sigma_{\mathcal{G}_1\cup\cdots\cup\mathcal{G}_m}^{(q_1,\dots,q_m)}(U) = \int_U \exp\bigg\{-\frac{i}{2}\sum_{k=1}^m Q_{m,(k)} \sum_{j_k=1}^{n_k} (u, e_{j_k}^{\mathcal{G}_k})_2^2\bigg\} d\sigma(u) \tag{5.2}$$

where $n_k = |\mathcal{G}_k|$ and

$$Q_{m,(k)} = \frac{1}{\alpha_m} - \frac{1}{q_k} \tag{5.3}$$

for each $k \in \{1,2,\dots,m\}$, and where $\alpha_m$ is given by (4.14). Then $\sigma_{\mathcal{G}_1\cup\cdots\cup\mathcal{G}_m}^{(q_1,\dots,q_m)}$ is an element of the sphere $\mathbb{S}(\sigma)$ defined by (4.3) above.

Given a functional $F_\sigma$ in $\mathcal{S}(L_2[0,T])$, let

$$\mathcal{R}_{\mathcal{G}_1\cup\cdots\cup\mathcal{G}_m}^{(q_1,\dots,q_m)}(F_\sigma) = F_{\sigma_{\mathcal{G}_1\cup\cdots\cup\mathcal{G}_m}^{(q_1,\dots,q_m)}}.$$

Then in view of (4.2), one can see that

$$\|F_\sigma\| = \big\|\mathcal{R}_{\mathcal{G}_1\cup\cdots\cup\mathcal{G}_m}^{(q_1,\dots,q_m)}(F_\sigma)\big\| \quad \text{i.e.,} \quad \|F_\sigma\| = \Big\|F_{\sigma_{\mathcal{G}_1\cup\cdots\cup\mathcal{G}_m}^{(q_1,\dots,q_m)}}\Big\|.$$

Under these conventions, we assert the following Fubini theorem for the iterated CFFT.

**Theorem 5.1.** *Given a finite subset $\mathcal{G}$ of $\mathcal{N}$, the partition $\{\mathcal{G}_k\}_{k=1}^m$ of $\mathcal{G}$ and the conditioning function $X_{\mathcal{G}}$ be as above. Then for any functional $F_\sigma \in \mathcal{S}(L_2[0,T])$ given by* (4.1) *and any finite sequence $Q = \{q_1,\dots,q_m\}$ in $\mathbb{R}\setminus\{0\}$ which satisfies the condition* (4.12), *it follows that*

$$\begin{aligned} &T_{q_m}^{(1)}\Big(T_{q_{m-1}}^{(1)}\Big(\cdots T_{q_1}^{(1)}(F_\sigma|X_{\mathcal{G}_1})(\,\cdot\,,\vec{\xi}^{(1)})\cdots\Big|X_{\mathcal{G}_{m-1}}\Big)(\,\cdot\,,\vec{\xi}^{(m-1)})\Big|X_{\mathcal{G}_m}\Big)(y,\vec{\xi}^{(m)}) \\ &= T_{\alpha_m}^{(1)}\Big(F_{\sigma_{\mathcal{G}_1\cup\cdots\cup\mathcal{G}_m}^{(q_1,\dots,q_m)}}\Big|X_{\mathcal{G}}\Big)(y,\vec{\xi}^{(1)}\wedge\cdots\wedge\vec{\xi}^{(m)}) \end{aligned} \tag{5.4}$$

*for SI-a.e. $y \in C_0[0,T]$ and a.e.*

$$\vec{\xi}^{(1)} \wedge \cdots \wedge \vec{\xi}^{(m)} = (\xi_1^{(1)}, \dots, \xi_{n_1}^{(1)}, \dots, \xi_1^{(m)}, \dots, \xi_{n_m}^{(m)}) \in \mathbb{R}^{n_1+\cdots+n_m},$$

*where $\alpha_m$ is given by* (4.14)*. Also, both of the expressions in* (5.4) *are given by the expression*

$$\int_{L_2[0,T]} \exp\bigg\{ i\langle u, y\rangle - \frac{i}{2\alpha_m}\bigg[\|u\|_2^2 - \sum_{j=1}^{n}(u, e_j^{\mathcal{G}})_2^2\bigg] + i\sum_{k=1}^{m}\sum_{j=1}^{n}\xi_j^{(k)}(u, e_j^{\mathcal{G}})_2\bigg\} d\sigma(u).$$

*Proof.* First, in view of Remark 4.5, the iterated CFFT in (5.4) exists. Next, using (4.7) $m$-times, (4.14), (5.3), (5.2), and (4.7) again, it follows that

$$\begin{aligned}
&T_{q_m}^{(1)}\Big(T_{q_{m-1}}^{(1)}\Big(\cdots T_{q_1}^{(1)}(F_\sigma|X_{\mathcal{G}_1})(\,\cdot\,, \vec{\xi}^{(1)})\cdots\Big|X_{\mathcal{G}_{m-1}}\Big)(\,\cdot\,, \vec{\xi}^{(m-1)})\Big|X_{\mathcal{G}_m}\Big)(y, \vec{\xi}^{(m)}) \\
&= \int_{L_2[0,T]} \exp\bigg\{ i\langle u, y\rangle - \sum_{k=1}^{m}\frac{i}{2q_k}\bigg[\|u\|_2^2 - \sum_{j_k=1}^{n_k}(u, e_{j_k}^{\mathcal{G}_k})_2^2\bigg] + i\sum_{k=1}^{m}\sum_{j_k=1}^{n_k}\xi_{j_k}^{(k)}(u, e_{j_k}^{\mathcal{G}_k})_2\bigg\} d\sigma(u) \\
&= \int_{L_2[0,T]} \exp\bigg\{ i\langle u, y\rangle - \frac{i}{2\alpha_m}\|u\|_2^2 + i\sum_{k=1}^{m}\frac{1}{2q_k}\sum_{j_k=1}^{n_k}(u, e_{j_k}^{\mathcal{G}_k})_2^2 \\
&\qquad\qquad + i\sum_{k=1}^{m}\sum_{j_k=1}^{n_k}\xi_{j_k}^{(k)}(u, e_{j_k}^{\mathcal{G}_k})_2\bigg\} d\sigma(u) \\
&= \int_{L_2[0,T]} \exp\bigg\{ i\langle u, y\rangle - \frac{i}{2\alpha_m}\bigg[\|u\|_2^2 - \sum_{k=1}^{m}\sum_{j_k=1}^{n_k}(u, e_{j_k}^{\mathcal{G}_k})_2^2\bigg] + i\sum_{k=1}^{m}\sum_{j_k=1}^{n_k}\xi_{j_k}^{(k)}(u, e_{j_k}^{\mathcal{G}_k})_2 \\
&\qquad\qquad - \frac{i}{2\alpha_m}\sum_{k=1}^{m}\sum_{j_k=1}^{n_k}(u, e_{j_k}^{\mathcal{G}_k})_2^2 + i\sum_{k=1}^{m}\frac{1}{2q_k}\sum_{j_k=1}^{n_k}(u, e_{j_k}^{\mathcal{G}_k})_2^2\bigg\} d\sigma(u) \\
&= \int_{L_2[0,T]} \exp\bigg\{ i\langle u, y\rangle - \frac{i}{2\alpha_m}\bigg[\|u\|_2^2 - \sum_{j=1}^{n_1+\cdots+n_m}(u, e_j^{\mathcal{G}_1\cup\cdots\cup\mathcal{G}_m})_2^2\bigg] \\
&\qquad\qquad + i\sum_{k=1}^{m}\sum_{j_k=1}^{n_k}\xi_{j_k}^{(k)}(u, e_{j_k}^{\mathcal{G}_k})_2 - \frac{i}{2}\sum_{k=1}^{m}Q_{m,(k)}\sum_{j_k=1}^{n_k}(u, e_{j_k}^{\mathcal{G}_k})_2^2\bigg\} d\sigma(u) \\
&= \int_{L_2[0,T]} \exp\bigg\{ i\langle u, y\rangle - \frac{i}{2\alpha_m}\bigg[\|u\|_2^2 - \sum_{j=1}^{n_1+\cdots+n_m}(u, e_j^{\mathcal{G}_1\cup\cdots\cup\mathcal{G}_m})_2^2\bigg] \\
&\qquad\qquad + i\sum_{k=1}^{m}\sum_{j_k=1}^{n_k}\xi_{j_k}^{(k)}(u, e_{j_k}^{\mathcal{G}_k})_2\bigg\} d\sigma_{\mathcal{G}_1\cup\cdots\cup\mathcal{G}_m}^{(q_1,\dots,q_m)}(u) \\
&= T_{\alpha_m}^{(1)}\Big(F_{\sigma_{\mathcal{G}_1\cup\cdots\cup\mathcal{G}_m}^{(q_1,\dots,q_m)}}\Big|X_{\mathcal{G}}\Big)(y, \vec{\xi}^{(1)} \wedge \cdots \wedge \vec{\xi}^{(m)})
\end{aligned}$$

for SI-a.e. $y \in C_0[0,T]$ and a.e.

$$\vec{\xi}^{(1)} \wedge \cdots \wedge \vec{\xi}^{(m)} = (\xi_1^{(1)}, \dots, \xi_{n_1}^{(1)}, \dots, \xi_1^{(m)}, \dots, \xi_{n_m}^{(m)}) \in \mathbb{R}^{n_1+\cdots+n_m},$$

as desired. □

We provide another Fubini theorem for the iterated CFFT without proof.

**Theorem 5.2.** *Let $\mathcal{G}$, $\{\mathcal{G}_k\}_{k=1}^m$, and $X_{\mathcal{G}}$ be as in Theorem 5.1. Then for any functional $F_\sigma \in \mathcal{S}(L_2[0,T])$ given by* (4.1) *and any finite sequence $Q = \{q_1, \ldots, q_m\}$ in $\mathbb{R} \setminus \{0\}$ which satisfies the condition* (4.12)*, it follows that*

$$\begin{aligned} &T_{q_m}^{(1)}\Big(T_{q_{m-1}}^{(1)}\Big(\cdots T_{q_1}^{(1)}\Big(F_{\sigma_{\mathcal{G}_1\cup\cdots\cup\mathcal{G}_m}^{(-q_1,\ldots,-q_m)}}\Big|X_{\mathcal{G}_1}\Big)(\,\cdot\,,\vec{\xi}^{(1)})\cdots\Big|X_{\mathcal{G}_{m-1}}\Big)(\,\cdot\,,\vec{\xi}^{(m-1)})\Big|X_{\mathcal{G}_m}\Big)(y,\vec{\xi}^{(m)}) \\ &= T_{\alpha_m}^{(1)}\big(F_\sigma\big|X_{\mathcal{G}}\big)\big(y,\vec{\xi}^{(1)}\wedge\cdots\wedge\vec{\xi}^{(m)}\big) \end{aligned}$$

*for SI-a.e. $y \in C_0[0,T]$ and a.e.*

$$\vec{\xi}^{(1)}\wedge\cdots\wedge\vec{\xi}^{(m)} = (\xi_1^{(1)},\ldots,\xi_{n_1}^{(1)},\ldots,\xi_1^{(m)},\ldots,\xi_{n_m}^{(m)}) \in \mathbb{R}^{n_1+\cdots+n_m},$$

*where $\alpha_m$ is given by* (4.14)*.*

We finish this paper with Fubini theorems for iterated conditional Feynman integrals.

**Corollary 5.3.** *Let $\mathcal{G}$, $\{\mathcal{G}_k\}_{k=1}^m$, and $X_{\mathcal{G}}$ be as in Theorem 5.1. Then for any functional $F_\sigma \in \mathcal{S}(L_2[0,T])$ given by* (4.1) *and any finite sequence $Q = \{q_1, \ldots, q_m\}$ in $\mathbb{R} \setminus \{0\}$ which satisfies the condition* (4.12)*, it follows that*

$$\begin{aligned} &E^{\mathrm{anf}_{q_m}}\Big(E^{\mathrm{anf}_{q_{m-1}}}\Big(\cdots E^{\mathrm{anf}_{q_1}}\Big(F_\sigma\circ A_m(x_1,\ldots,x_m)\Big|X_{\mathcal{G}_1}(x_1)=\vec{\xi}^{(1)}\Big) \\ &\qquad\qquad\cdots\Big|X_{\mathcal{G}_{m-1}}(x_{m-1})=\vec{\xi}^{(m-1)}\Big)\Big|X_{\mathcal{G}_m}(x_m)=\vec{\xi}^{(m)}\Big) \\ &= E^{\mathrm{anf}_{\alpha_m}}\Big(F_{\sigma_{\mathcal{G}_1\cup\cdots\cup\mathcal{G}_m}^{(q_1,\ldots,q_m)}}(x)\Big|X_{\mathcal{G}}(x)=\vec{\xi}^{(1)}\wedge\cdots\wedge\vec{\xi}^{(m)}\Big) \end{aligned}$$

*and*

$$\begin{aligned} &E^{\mathrm{anf}_{q_m}}\Big(E^{\mathrm{anf}_{q_{m-1}}}\Big(\cdots E^{\mathrm{anf}_{q_1}}\Big(F_{\sigma_{\mathcal{G}_1\cup\cdots\cup\mathcal{G}_m}^{(-q_1,\ldots,-q_m)}}\circ A_m(x_1,\ldots,x_m)\Big|X_{\mathcal{G}_1}(x_1)=\vec{\xi}^{(1)}\Big) \\ &\qquad\qquad\cdots\Big|X_{\mathcal{G}_{m-1}}(x_{m-1})=\vec{\xi}^{(m-1)}\Big)\Big|X_{\mathcal{G}_m}(x_m)=\vec{\xi}^{(m)}\Big) \\ &= E^{\mathrm{anf}_{\alpha_m}}\Big(F_\sigma(x)\Big|X_{\mathcal{G}}(x)=\vec{\xi}^{(1)}\wedge\cdots\wedge\vec{\xi}^{(m)}\Big) \end{aligned}$$

*for SI-a.e. $y \in C_0[0,T]$ and a.e.*

$$\vec{\xi}^{(1)}\wedge\cdots\wedge\vec{\xi}^{(m)} = (\xi_1^{(1)},\ldots,\xi_{n_1}^{(1)},\ldots,\xi_1^{(m)},\ldots,\xi_{n_m}^{(m)}) \in \mathbb{R}^{n_1+\cdots+n_m},$$

*respectively, where $\alpha_m$ is given by* (4.14)*.*

**An epilogue** In the celebrated paper [25], Park and and Skoug suggested the concept of the CFFT and the corresponding conditional convolution product of SIM functionals on $C_0[0,T]$. These fundamental concepts have been very useful to us in establishing many of the results in [8, 9]. We feel strongly that the fundamental concepts in [25] will prove to be very useful in future work by ourselves as well as other researchers in this area. The framework and methods we used to obtain the results in this article are very dependent upon the idea in the paper [24] concerning the evaluation formula for conditional Wiener integral given vector-valued conditioning function.

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