Bull. Math. Soc. Sci. Math. Roumanie
Tome 68 (116), No. 2, 2025, 127–148

Counting elementary moves in the optimal solution of the Tower of Hanoi problem

by
Hacène Belbachir[(1)], El-Mehdi Mehiri[(2)]

Abstract

In the Tower of Hanoi puzzle, moving a disc from one peg to another is called an elementary move, in total there are six elementary moves. In this paper, we present the sequence φ_n, and how it can be applied to find the numbers $f_n^{ij}(x)$, respectively $g_n^{ij}(d,x)$, of moves of one of six types x made by all, respectively each, of the n discs d in the optimal solution for the classical Tower of Hanoi game to transfer a tower from peg i to peg j. We establish many results related to φ_n, $f_n^{ij}(x)$, and $g_n^{ij}(d,x)$, such as explicit and implicit forms, generating functions, and more.

Key Words: Tower of Hanoi, recursion, sequences, enumeration, counting.
2020 Mathematics Subject Classification: 00A08, 05A15, 05A19, 11Y55.

1 Introduction

The problem of the Tower of Hanoi is one of the most famous problems used to introduce the concept of mathematical induction. Since its invention in 1883 by the French mathematician Édouard Lucas [13, p. 55–59], this game has received a lot of attention from mathematicians due to the interesting mathematics hiding in and around this puzzle.

Recall that the Tower of Hanoi puzzle consists of n discs of different sizes, and three pegs i, j, and k. In the beginning, all discs are stacked on one of the three pegs, in which no disc lies on top of a smaller disc. The goal is to transfer the whole tower of the n discs to another peg using the minimum number of moves, where a legal move is to move one topmost disc at a time and never put a disc on a smaller one.

The Tower of Hanoi with n discs can be solved optimally in $u_n = 2^n - 1$ moves, using the following recursive procedure: First move the sub-tower of the first $(n-1)$ discs from the source peg i to the auxiliary peg k where j is the destination peg, then move the biggest disc to the destination peg j, and finally move the sub-tower of the first $(n-1)$ discs to the final peg. This recursive procedure satisfies the following recurrence relation

$$u_0 = 0, \quad u_n = 2u_{n-1} + 1, \quad n \geq 1. \tag{1.1}$$

The optimal sequence of moves is unique and is of length $2^n - 1$ [8, Theorem 2.1]. We call an elementary move x, a move of a disc from one peg to another peg. There are six elementary moves $a = ij$ (which denotes a move of a topmost disc from peg i to peg j), $b = jk$, $c = ki$, $\overline{a} = ji$, $\overline{b} = kj$, and $\overline{c} = ik$. This coding of moves is inspired from [1].

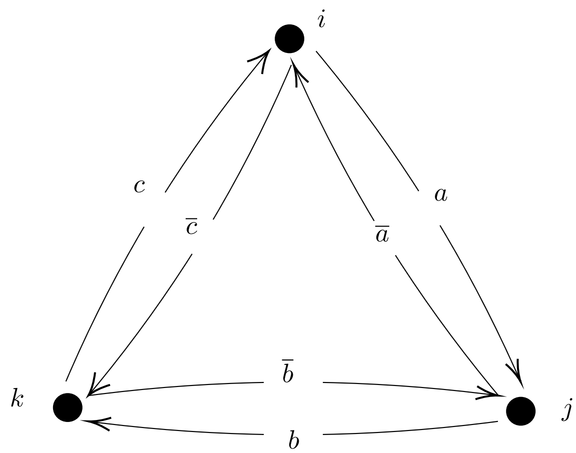

Figure 1: A complete digraph with three vertices which represents the three pegs and the arcs represent the moves between pegs.

In this paper, we introduce a recursive function that counts the number of each elementary move in the optimal sequence of moves. We denote this function by $f_n^{ij}(x)$ where $x \in \mathcal{A} = \{a, b, c, \overline{a}, \overline{b}, \overline{c}\}$ is the elementary move to be counted in the optimal sequence of moves that transfer a tower of n discs from peg i to peg j.

$$f_n^{ij} : \begin{cases} \mathcal{A} \to \mathbb{N}; \\ x \mapsto f_n^{ij}(x). \end{cases}$$

We also introduce another recursive function

$$g_n^{ij} : \begin{cases} \mathcal{D} \times \mathcal{A} \to \mathbb{N}; \\ (d, x) \mapsto g_n^{ij}(d, x), \end{cases}$$

where $\mathcal{D} = \{1, \ldots, n\}$ is the set of discs numbered in increasing order of size. This function calculates the number of each elementary move that each disc makes during the optimal solution, i.e., given a specified disc $d \in \mathcal{D}$ and an elementary move $x \in \mathcal{A}$, then $g_n^{ij}(d, x)$ is the number of times disc d made the elementary move x during the optimal solution that transfers a tower of n discs from peg i to peg j.

In this paper, we have employed Iverson's convention which is defined by

$$[S] = \begin{cases} 1, & \text{if } S \text{ is true;} \\ 0, & \text{otherwise,} \end{cases}$$

where S be a mathematical statement.

We present combinatorial results and properties of these two functions such as recurrent relations, explicit and implicit formulas, ordinary generating functions, and more.

But before starting to deal with these functions, we present the sequence φ_n, which will be very useful when dealing with the functions $f_n^{ij}(x)$ and $g_n^{ij}(d, x)$, it will be related to these two functions to obtain many of their properties. The presentation of the sequence φ_n will be the main topic of the next section.

In [2, Section 6.4], the authors showed that there is a 6-state finite automaton computing the nth move of the optimal solution to the Tower of Hanoi puzzle, on input n in base 2, and therefore one can compute the number of each type of move by considering the matrices that encode the transitions between states on inputs 0 and 1. This is more or less the same output of our function $f_n^{ij}(x)$ which is the number of each elementary move in the optimal sequence of moves. Also, using the facts in [8, Chapter 2], one can find the numbers $f_n^{ij}(x)$, respectively $g_n^{ij}(d,x)$, of moves of one of six types x made by all, respectively each, of the n discs d in the optimal solution for the classical Tower of Hanoi game to transfer a tower from peg i to peg j, as we will see in Section 4. Since each disc 1 moves exactly 2^{n-1} times in a cyclic way [8, Proposition 2.4], and observing that the Lichtenberg sequence l can be characterized by $l_{n-1} = \frac{1}{3}(2^n - 2^{n \mod 2})$ (cf. [6] for more about the Lichtenberg sequence), then we arrive, for odd n, at $g_n^{ij}(1,a) = l_{n-2}+1$ and $g_n^{ij}(1,x) = l_{n-2}$ if $x \in \{b,c\}$; similarly, $g_n^{ij}(1,\overline{a}) = l_{n-2}$ and $g_n^{ij}(1,\overline{x}) = l_{n-2}+1$ if $x \in \{b,c\}$ for even $n \geq 1$. All other values of $g_n^{ij}(1,x)$ are 0. This summarizes essentially the content of Remark 3. Making use of Proposition 6 we obtain all the numbers $g_n^{ij}(d,x)$ and summing over d, i.e., using identity (4.2), we arrive at the results on $f_n^{ij}(x)$ of Remark 2.

We shall mention that many classical sequences are hidden in the Tower of Hanoi puzzle and in its variations. We mention here Stern's diatomic sequence [7], the Stirling numbers of the second kind [11], the second order Eulerian numbers, Lah numbers and Catalan numbers [12], Fibonacci numbers [10], the Anti-Ramsey numbers [4], but also Sierpinski gasket [9] and the Pascal triangle [5]. We mention also [3] for more counting on the Tower of Hanoi. For more information about the Tower of Hanoi problem, we refer to the comprehensive monograph [8].

2 Presentation of an interesting sequence

In this section we present a new integer sequence that is closely related to the Tower of Hanoi as it is described in the next section, we denote this sequence by φ_n. We show that in the optimal sequence of moves, the number of elementary moves a is $\varphi_n - 3\varphi_{n-2}$, while elementary moves b and c appear the same number of times which is φ_{n-2}; on the other side the number of elementary move $\overline{a}$ is $2\varphi_{n-3}$, and the number of elementary move $\overline{b}$ is the same as the number of elementary move $\overline{c}$ which equals to $\varphi_{n-1} - 2\varphi_{n-3}$. As we can see the sequence φ_n appears in the number of each elementary move. Therefore, before we present the recursive function $f_n^{ij}(x)$ mentioned in the introduction, we present some properties and combinatorial identities of the sequence φ_n.

Definition 1. *For all integers $n \geq 2$, we define φ_n using the following recurrence relation*

$$\varphi_n = 4\varphi_{n-2} + \left\lfloor \frac{n+1}{2} \right\rfloor, \tag{2.1}$$

with $\varphi_0 = 0$, $\varphi_1 = 1$.

Property 1. *For all integers $n \geq 1$, we have*

$$\varphi_{2n-1} = \varphi_{2n}.$$

Proof. For $n = 1$, we have $\varphi_1 = \varphi_2 = 1$. Now we suppose that the property is true up to $n - 1 \geq 1$, then we have

$$\varphi_{2n-1} = 4\varphi_{2n-3} + \left\lfloor \frac{2n}{2} \right\rfloor = 4\varphi_{2n-3} + n,$$
$$\varphi_{2n} = 4\varphi_{2n-2} + \left\lfloor \frac{2n+1}{2} \right\rfloor = 4\varphi_{2n-2} + n.$$

By the induction hypothesis we have $\varphi_{2n-3} = \varphi_{2n-2}$. Hence the result. □

Remark 1. *For all integers $n \geq 1$, we have*

$$\varphi_{2n-1} = \varphi_{2n} = A014825(n).$$

The sequence φ_n does not yet have a combinatorial interpretation nor does the sequence $A014825$. However, we will see in the next sections, that this sequence is related to f_n^{ij} and g_n^{ij}. The first few terms of sequence φ_{2n} are

$$0, 1, 6, 27, 112, 453, 1818, 7279, 29124, 116505, \ldots$$

Let us mention that the sequence $(\varphi_{2n})_{n\geq 0}$ is the sequence of partial sums of the odd Lichtenberg numbers

$$m_0 = 0, \quad m_n = l_{2n-1}, \quad n \geq 1,$$

cf. [6], where

$$l_n = \frac{2}{3} \cdot 2^n - \frac{1}{6}(-1)^n - \frac{1}{2}, \quad n \geq 0,$$

is the Lichtenberg sequence A000975 in the OEIS.

Moreover, the odd Lichtenberg numbers $m_n = \frac{1}{3}(4^n - 1)$, $n \geq 0$, are A002450, and the even Lichtenberg numbers given by $l_{2n} = 2l_{2n-1} = 2m_n$, $n \geq 0$, are A020988 with partial sums A145766 in the OEIS (note that the sequences of Lichtenberg numbers appear later in Remark 3).

In the rest of this section, we establish some of the properties of the sequence φ such as explicit formulas and ordinary generating functions.

Lemma 1. *For all integers $n \geq 4$, we have*

$$\varphi_n = 5\varphi_{n-2} - 4\varphi_{n-4} + 1, \tag{2.2}$$

with $\varphi_0 = 0$, $\varphi_1 = \varphi_2 = 1$, and $\varphi_3 = 6$.

Proof. We have $\varphi_n = 4\varphi_{n-2} + \lfloor \frac{n+1}{2} \rfloor$ and $\varphi_{n-2} = 4\varphi_{n-4} + \lfloor \frac{n-1}{2} \rfloor$, then

$$\varphi_n - \varphi_{n-2} = 4\varphi_{n-2} - 4\varphi_{n-4} + \left\lfloor \frac{n+1}{2} \right\rfloor - \left\lfloor \frac{n-1}{2} \right\rfloor.$$

We have $\lfloor \frac{n+1}{2} \rfloor - \lfloor \frac{n-1}{2} \rfloor = 1$ for all $n \geq 0$, therefore $\varphi_n = 5\varphi_{n-2} - 4\varphi_{n-4} + 1$. □

Theorem 1. *Let $P(z) = \sum_{n\geq 0} \varphi_n z^n$ be the ordinary generating function of φ_n, then*

$$P(z) = \frac{z}{(1-4z^2)(1+z)(1-z)^2}. \tag{2.3}$$

Proof. We have $\varphi_n = 5\varphi_{n-2} - 4\varphi_{n-4} + 1$ then

$$\sum_{n\geq 4} \varphi_n z^{n-4} = 5\sum_{n\geq 4} \varphi_{n-2} z^{n-4} - 4\sum_{n\geq 4} \varphi_{n-4} z^{n-4} + \sum_{n\geq 4} z^{n-4},$$

therefore

$$z^{-4}(P(z) - \varphi_0 - z\varphi_1 - z^2\varphi_2 - z^3\varphi_3) = 5z^{-2}(P(z) - \varphi_0 - z\varphi_1) - 4P(z) + \frac{1}{1-z}.$$

Hence

$$P(z)(1 - 5z^2 + 4z^4) = z + z^2 + z^3 + \frac{z^4}{1-z}.$$

Finally

$$P(z) = \frac{z}{(1-4z^2)(1+z)(1-z)^2}.$$ ∎

Corollary 1. *For all integers $n \geq 0$, we have*

$$\varphi_n = \sum_{r+s+2t=n} (-1)^r s 4^t. \tag{2.4}$$

Proof. We have

$$\begin{aligned} P(z) &= \sum_{n\geq 0} \varphi_n z^n \\ &= \sum_{r\geq 0} (-1)^r z^r \sum_{s\geq 0} s z^s \sum_{t\geq 0} 4^t z^{2t} \\ &= \sum_{n\geq 0} \left(\sum_{r+s+2t=n} (-1)^r s 4^t \right) z^n. \end{aligned}$$

It finally comes by identification

$$\varphi_n = \sum_{r+s+2t=n} (-1)^r s 4^t.$$ ∎

Corollary 2. *For all integers $n \geq 0$, we have*

$$\varphi_n = \sum_{t=0}^{\lfloor \frac{n}{2} \rfloor} 4^t \left\lfloor \frac{n-2t+1}{2} \right\rfloor. \tag{2.5}$$

Proof. We have

$$\begin{aligned}
\varphi_n &= \sum_{r+s+2t=n} (-1)^r s 4^t \\
&= \sum_{t=0}^{\lfloor \frac{n}{2} \rfloor} 4^t \sum_{r+s=n-2t} (-1)^r s \\
&= \sum_{t=0}^{\lfloor \frac{n}{2} \rfloor} 4^t \sum_{s=0}^{n-2t} (-1)^{n-2t-s} s \\
&= \sum_{t=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^{n-2t} 4^t \sum_{s=0}^{n-2t} (-1)^s s.
\end{aligned}$$

We know that

$$\sum_{s=0}^{n} (-1)^s s = (-1)^n \left\lfloor \frac{n+1}{2} \right\rfloor = \begin{cases} \frac{n}{2}, & \text{if } n \text{ even;} \\ -\frac{n+1}{2}, & \text{otherwise.} \end{cases}$$

Therefore

$$\begin{aligned}
\varphi_n &= \sum_{t=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^{n-2t} 4^t (-1)^{n-2t} \left\lfloor \frac{n-2t+1}{2} \right\rfloor \\
&= \sum_{t=0}^{\lfloor \frac{n}{2} \rfloor} 4^t \left\lfloor \frac{n-2t+1}{2} \right\rfloor.
\end{aligned}$$

∎

Lemma 2. *For all integers $n \geq 0$, we have*

$$\sum_{t=0}^{n} t4^t = \frac{4}{9}(4^n(3n-1)+1). \tag{2.6}$$

Proof. For $n = 0$, we have $\sum\limits_{t=0}^{0} t4^t = 0 = \frac{4}{9}(4^0(3(0)-1)+1)$. Let us suppose that the lemma is true up to $n-1$, and we will prove it for n. We have $\sum\limits_{t=0}^{n} t4^t = \sum\limits_{t=0}^{n-1} 4^t t + n4^n = \frac{4}{9}(4^{n-1}(3(n-1)-1)+1) + n4^n = \frac{4}{9}(3n4^{n-1} - 4^n + 1 + 9n4^{n-1}) = \frac{4}{9}(4^n(3n-1)+1)$. ∎

Corollary 3. *For all integers $n \geq 0$, we have*

$$\varphi_n = \frac{1}{9}(4^{\lfloor \frac{n+1}{2} \rfloor + 1} - 3\left\lfloor \frac{n+1}{2} \right\rfloor - 4) = \begin{cases} \frac{1}{18}(2^{n+3} - 3n - 8), & \textit{if } n \textit{ even;} \\ \frac{1}{18}(2^{n+4} - 3n - 11), & \textit{otherwise.} \end{cases} \tag{2.7}$$

and $\varphi_{-1} = 0$.

Proof. We have $\varphi_n = \sum_{t=0}^{\lfloor \frac{n}{2} \rfloor} 4^t \left\lfloor \frac{n-2t+1}{2} \right\rfloor$, then

$$\varphi_n = \sum_{t=0}^{\lfloor \frac{n}{2} \rfloor} 4^t \frac{n-2t}{2} = \frac{1}{2}\left(n \sum_{t=0}^{\frac{n}{2}} 4^t - 2 \sum_{t=0}^{\frac{n}{2}} 4^t t \right), \qquad \text{if } n \text{ even.}$$
$$\varphi_n = \sum_{t=0}^{\lfloor \frac{n-1}{2} \rfloor} 4^t \frac{n-2t+1}{2} = \frac{1}{2}\left((n+1) \sum_{t=0}^{\frac{n-1}{2}} 4^t - 2 \sum_{t=0}^{\frac{n-1}{2}} 4^t t \right), \qquad \text{if } n \text{ odd.}$$

Using Lemma 2 and some elementary calculations we can find the result. ∎

Corollary 4. *For all integers $n \geq 0$, we have*

$$\varphi_n = \sum_{t=0}^{\lfloor \frac{n+1}{2} \rfloor - 1} \binom{\lfloor \frac{n+1}{2} \rfloor + 1}{t} 3^t. \tag{2.8}$$

Proof. We have

$$\begin{aligned}
\varphi_n &= \frac{1}{9}(4^{\lfloor \frac{n+1}{2} \rfloor + 1} - 3 \left\lfloor \frac{n+1}{2} \right\rfloor - 4) \\
&= \frac{1}{9}(\sum_{t=0}^{\lfloor \frac{n+1}{2} \rfloor + 1} \binom{\lfloor \frac{n+1}{2} \rfloor + 1}{t} 3^t - 3 \left\lfloor \frac{n+1}{2} \right\rfloor - 4) \\
&= \frac{1}{9}(\sum_{t=2}^{\lfloor \frac{n+1}{2} \rfloor + 1} \binom{\lfloor \frac{n+1}{2} \rfloor + 1}{t} 3^t + 1 + 3(\left\lfloor \frac{n+1}{2} \right\rfloor + 1) - 3 \left\lfloor \frac{n+1}{2} \right\rfloor - 4) \\
&= \sum_{t=2}^{\lfloor \frac{n+1}{2} \rfloor + 1} \binom{\lfloor \frac{n+1}{2} \rfloor + 1}{t} 3^{t-2} \\
&= \sum_{t=0}^{\lfloor \frac{n+1}{2} \rfloor - 1} \binom{\lfloor \frac{n+1}{2} \rfloor + 1}{t} 3^t.
\end{aligned}$$
∎

Corollary 5. *For all integers $n \geq 0$, we have*

$$\varphi_n = \sum_{t=0}^{\lfloor \frac{n+1}{2} \rfloor} \sum_{r=0}^{2t} (-1)^{r+1} J_r J_{2t-r}, \tag{2.9}$$

where J_n is the Jacobsthal sequence.

Proof. We know that $J_n = \frac{1}{3}(2^n - (-1)^n)$ [14], then

$$\begin{aligned}
\sum_{t=0}^{\lfloor \frac{n+1}{2} \rfloor} \sum_{r=0}^{2t} (-1)^{r+1} J_r J_{2t-r} &= \frac{1}{9} \sum_{t=0}^{\lfloor \frac{n+1}{2} \rfloor} \sum_{r=0}^{2t} (-1)^{r+1} (2^r - (-1)^r)(2^{2t-r} - (-1)^{2t-r}) \\
&= \frac{1}{9} \sum_{t=0}^{\lfloor \frac{n+1}{2} \rfloor} \sum_{r=0}^{2t} (-1)^{r+1} (2^{2t} + (-1)^{r+1} 2^r + (-1)^{r+1} 2^{2t-r} + 1) \\
&= \frac{1}{9} \sum_{t=0}^{\lfloor \frac{n+1}{2} \rfloor} \left(\sum_{r=0}^{2t} (2^r + 2^{2t-r} + (-1)^{r+1}(2^{2t}+1)) \right) \\
&= \frac{1}{9} \sum_{t=0}^{\lfloor \frac{n+1}{2} \rfloor} \sum_{r=0}^{2t} 2^r + 2^{2t} \sum_{r=0}^{2t} \frac{1}{2^r} - (2^{2t}+1) \sum_{r=0}^{2t} (-1)^r \\
&= \frac{1}{9} \sum_{t=0}^{\lfloor \frac{n+1}{2} \rfloor} (2^{2t+1} - 1 + 2^{2t+1} - 1 - 2^{2t} - 1) \\
&= \frac{1}{9} \sum_{t=0}^{\lfloor \frac{n+1}{2} \rfloor} (3 \times 4^t - 3) \\
&= \frac{1}{9} (4^{\lfloor \frac{n+1}{2} \rfloor + 1} - 1 - 3(\left\lfloor \frac{n+1}{2} \right\rfloor + 1)) \\
&= \frac{1}{9} (4^{\lfloor \frac{n+1}{2} \rfloor + 1} - 3 \left\lfloor \frac{n+1}{2} \right\rfloor - 4) \\
&= \varphi_n.
\end{aligned}$$

□

3 Counting the number of each elementary move

In this section, we present the recursive function

$$f_n^{ij} : \begin{cases} \mathcal{A} \to \mathbb{N}; \\ x \mapsto f_n^{ij}(x), \end{cases}$$

where $x \in \mathcal{A} = \{a, b, c, \overline{a}, \overline{b}, \overline{c}\}$ is an elementary move. $f_n^{ij}(x)$ counts the number of times that the move x appears in the optimal sequence of elementary moves that transfers a tower of n discs from peg i to peg j using peg k as an auxiliary peg. The optimal sequences are as follows:

n	optimal sequence
1	a
2	$\bar{c}\,a\,\bar{b}$
3	$a\,\bar{c}\,b\,a\,c\,\bar{b}\,a$
4	$\bar{c}\,a\,\bar{b}\,\bar{c}\,\bar{a}\,b\,\bar{c}\,a\,\bar{b}\,c\,\bar{a}\,\bar{b}\,\bar{c}\,a\,\bar{b}$
5	$a\,\bar{c}\,b\,a\,c\,\bar{b}\,a\,\bar{c}\,b\,\bar{a}\,c\,b\,a\,\bar{c}\,b\,a\,c\,\bar{b}\,a\,c\,b\,\bar{a}\,c\,\bar{b}\,a\,\bar{c}\,b\,a\,c\,\bar{b}\,a$
6	$\bar{c}\,a\,\bar{b}\,\bar{c}\,\bar{a}\,b\,\bar{c}\,a\,\bar{b}\,c\,\bar{a}\,\bar{b}\,\bar{c}\,a\,\bar{b}\,\bar{c}\,\bar{a}\,b\,\bar{c}\,\bar{a}\,\bar{b}\,c\,\bar{a}\,b\,\bar{c}\,a\,\bar{b}\,\bar{c}\,\bar{a}\,b\,\bar{c}\,a\,\bar{b}\,c\,\bar{a}\,\bar{b}\,\bar{c}\,a\,\bar{b}\,c\,\bar{a}\,b\,\bar{c}\,\bar{a}\,\bar{b}\,c\,\bar{a}\,\bar{b}\,\bar{c}$ $a\,\bar{b}\,\bar{c}\,\bar{a}\,b\,\bar{c}\,a\,\bar{b}\,c\,\bar{a}\,\bar{b}\,\bar{c}\,a\,\bar{b}$

Table 1: The optimal sequences of moves for the first values of n.

The following table shows the number of elementary moves in the optimal sequences above.

n	$f_n^{ij}(a)$	$f_n^{ij}(\bar{c})$	$f_n^{ij}(\bar{b})$	$f_n^{ij}(b)$	$f_n^{ij}(c)$	$f_n^{ij}(\bar{a})$	$\sum_{x\in\mathcal{A}} f_n^{ij}(x)$
1	1	0	0	0	0	0	1
2	1	1	1	0	0	0	3
3	3	1	1	1	1	0	7
4	3	4	4	1	1	2	15
5	9	4	4	6	6	2	31
6	9	15	15	6	6	12	63
7	31	15	15	27	27	12	127
8	31	58	58	27	27	54	255
9	117	58	58	112	112	54	511
10	117	229	229	112	112	224	1023
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 2: The number of each elementary move in the optimal sequence for the first values of n.

It is clear that for all $n \geq 0$, and $x \in \mathcal{A}$, we have

$$\sum_{x\in\mathcal{A}} f_n^{ij}(x) = 2^n - 1. \tag{3.1}$$

We introduced $f_n^{ij}(x)$ as a recursive function because of the following theorem.

Theorem 2. *For all integers $n \geq 4$, $x \in \mathcal{A}$, $f_n^{ij}(x)$ satisfies the following recurrence relation*

$$f_n^{ij}(x) = 5f_{n-2}^{ij}(x) - 4f_{n-4}^{ij}(x) + 2[x \in \{\bar{a}\}] + [x \in \{b, c\}] - [x \in \{\bar{c}, \bar{b}\}] - 2[x \in \{a\}], \tag{3.2}$$

where $f_0^{ij}(x) = 0$, $f_1^{ij}(x) = [x \in \{a\}]$, $f_2^{ij}(x) = [x \in \{a, \bar{c}, \bar{b}\}]$, and $f_3^{ij}(x) = 3[x \in \{a\}]+[x \in \{\bar{c}, b, c, \bar{b}\}]$.

Proof. Let S_n^{ij} be the optimal sequence of moves that transfer a tower of n discs from peg i to peg j; then using the recursive procedure that solves the Tower of Hanoi problem which is described in the introduction, we find

$$S_n^{ij} = S_{n-1}^{ik}[i \to j]S_{n-1}^{kj},$$

where $[i \to j]$ denotes a single move from peg i to j. Then we obtain the following recursive formula

$$\begin{aligned}
f_n^{ij}(x) &= f_{n-1}^{ik}(x) + [x \in \{a\}] + f_{n-1}^{kj}(x) \\
&= f_{n-2}^{ij}(x) + [x \in \{\overline{c}\}] + f_{n-2}^{jk}(x) + [x \in \{a\}] + f_{n-2}^{ki}(x) + [x \in \{\overline{b}\}] + f_{n-2}^{ij}(x) \\
&= 2f_{n-2}^{ij}(x) + f_{n-1}^{ji}(x) - [x \in \{\overline{a}\}] + [x \in \{\overline{c}\}] + [x \in \{a\}] + [x \in \{\overline{b}\}] \\
&= 2f_{n-2}^{ij}(x) + f_{n-1}^{ji}(x) - [x \in \{\overline{a}\}] + [x \in \{\overline{c}, a, \overline{b}\}].
\end{aligned}$$

Which gives

$$f_{n-1}^{ji}(x) = f_n^{ij}(x) - 2f_{n-2}^{ij}(x) + [x \in \{\overline{a}\}] - [x \in \{\overline{c}, a, \overline{b}\}]. \tag{3.3}$$

Thus, we have

$$f_n^{ji}(x) = f_{n+1}^{ij}(x) - 2f_{n-1}^{ij}(x) + [x \in \{\overline{a}\}] - [x \in \{\overline{c}, a, \overline{b}\}], \tag{3.4}$$

and

$$f_{n-2}^{ji}(x) = f_{n-1}^{ij}(x) - 2f_{n-3}^{ij}(x) + [x \in \{\overline{a}\}] - [x \in \{\overline{c}, a, \overline{b}\}]. \tag{3.5}$$

On the other hand, we have

$$f_n^{ji}(x) = 2f_{n-2}^{ji}(x) + f_{n-1}^{ij}(x) - [x \in \{a\}] + [x \in \{b, \overline{a}, c\}]. \tag{3.6}$$

By replacing (3.4) and (3.5) in (3.6), we obtain

$$f_{n+1}^{ij}(x) = 5f_{n-1}^{ij}(x) - 4f_{n-3}^{ij}(x) + 2[x \in \{\overline{a}\}] + [x \in \{b, c\}] - [x \in \{\overline{c}, \overline{b}\}] - 2[x \in \{a\}].$$

Hence, the result. ∎

Proposition 1. *In the optimal solution, the number of times we move discs from the destination (resp., source) peg j (resp., i) to the auxiliary peg k is equal to the number of times we move discs from the auxiliary peg k to the source (resp., destination) peg i (resp., j).*

We will see further in this section the reason why this proposition is true.

Corollary 6. *For all integers $n \geq 5$, $x \in \mathcal{A}$, $f_n^{ij}(x)$ satisfies the following homogeneous recurrence relation*

$$f_n^{ij}(x) = f_{n-1}^{ij}(x) + 5f_{n-2}^{ij}(x) - 5f_{n-3}^{ij}(x) - 4f_{n-4}^{ij}(x) + 4f_{n-5}^{ij}(x). \tag{3.7}$$

Proof. From Theorem 2, we have

$$f_{n-1}^{ij}(x) = 5f_{n-3}^{ij}(x) - 4f_{n-5}^{ij}(x) + 2[x \in \{\overline{a}\}] + [x \in \{b, c\}] - [x \in \{\overline{c}, \overline{b}\}] - 2[x \in \{a\}], \tag{3.8}$$

By subtracting (3.8) from (3.2), we obtain

$$f_n^{ij}(x) - f_{n-1}^{ij}(x) = 5f_{n-2}^{ij}(x) - 4f_{n-4}^{ij}(x) - 5f_{n-3}^{ij}(x) + 4f_{n-5}^{ij}(x). \tag{3.9}$$

Hence, the result. ∎

By replacing x with each possible move of the six elementary moves in relation (3.2), we find a recurrence relation to the number of each elementary move in the optimal solution as the following corollary presents.

Corollary 7. *For all integers $n \geq 0$, and $x \in \mathcal{A}$, we have*

$$f_{n+1}^{ij}(a) = \begin{cases} 0, & \text{if } n = 0; \\ 1, & \text{if } n = 1, 2; \\ 5f_{n-1}^{ij}(a) - 4f_{n-3}^{ij}(a) - 2, & \text{otherwise.} \end{cases} \tag{3.10}$$

$$f_{n+1}^{ij}(b) = \begin{cases} 0, & \text{if } n = 0, 1, 2; \\ 5f_{n-1}^{ij}(b) - 4f_{n-3}^{ij}(b) + 1, & \text{otherwise.} \end{cases} \tag{3.11}$$

$$f_{n+1}^{ij}(c) = \begin{cases} 0, & \text{if } n = 0, 1, 2; \\ 5f_{n-1}^{ij}(c) - 4f_{n-3}^{ij}(c) + 1, & \text{otherwise.} \end{cases} \tag{3.12}$$

$$f_{n+1}^{ij}(\overline{a}) = \begin{cases} 0, & \text{if } n = 0, 1, 2; \\ 5f_{n-1}^{ij}(\overline{a}) - 4f_{n-3}^{ij}(\overline{a}) + 2, & \text{otherwise.} \end{cases} \tag{3.13}$$

$$f_{n+1}^{ij}(\overline{b}) = \begin{cases} 0, & \text{if } n = 0, 1; \\ 1, & \text{if } n = 2; \\ 5f_{n-1}^{ij}(\overline{b}) - 4f_{n-3}^{ij}(\overline{b}) - 1, & \text{otherwise.} \end{cases} \tag{3.14}$$

$$f_{n+1}^{ij}(\overline{c}) = \begin{cases} 0, & \text{if } n = 0, 1; \\ 1, & \text{if } n = 2; \\ 5f_{n-1}^{ij}(\overline{c}) - 4f_{n-3}^{ij}(\overline{c}) - 1, & \text{otherwise.} \end{cases} \tag{3.15}$$

Using this last corollary, we remark that $f_n^{ij}(\overline{c}) = f_n^{ij}(\overline{b})$, and $f_n^{ij}(b) = f_n^{ij}(c)$. Now we can see why Proposition 1 is true.

Theorem 3. *For all $x \in \mathcal{A}$, let $G^{ij}(z,x) = \sum_{n\geq 0} f_n^{ij}(x) z^n$ be the ordinary generating function of the sequence $f_n^{ij}(x)$, then*

$$G^{ij}(z,x) = \frac{\begin{array}{r}[x \in \{a\}]z + [x \in \{\overline{c}, \overline{b}\}]z^2 + ([x \in \{b,c\}] - 3[x \in \{a\}])z^3 \\ + 2([x \in \{\overline{a}\}] - [x \in \{\overline{c}, \overline{b}\}])z^4\end{array}}{(1-4z^2)(1+z)(1-z)^2}. \tag{3.16}$$

Proof. We have

$$f_n^{ij}(x) = 5f_{n-2}^{ij}(x) - 4f_{n-4}^{ij}(x) + \epsilon(x),$$

where $\epsilon(x) = 2[x \in \{\overline{a}\}] + [x \in \{b\}] + [x \in \{c\}] - [x \in \{\overline{c}\}] - [x \in \{\overline{b}\}] - 2[x \in \{a\}]$. Therefore

$$\sum_{n\geq 4} f_n^{ij}(x) z^{n-4} = 5\sum_{n\geq 4} f_{n-2}^{ij}(x) z^{n-4} - 4\sum_{n\geq 4} f_{n-4}^{ij}(x) z^{n-4} + \epsilon(x)\sum_{n\geq 4} z^{n-4},$$

which gives

$$G^{ij}(z,x)(z^{-4}-5z^{-2}+4)=z^{-4}(f_1^{ij}(x)z+f_2^{ij}(x)z^2+f_3^{ij}(x)z^3)-5z^{-2}(f_1^{ij}(x)z)+\frac{\epsilon(x)}{1-z},$$

then

$$G^{ij}(z,x)=\frac{\begin{array}{r}f_1^{ij}(x)z+(f_2^{ij}(x)-f_1^{ij}(x))z^2+(f_3^{ij}(x)-f_2^{ij}(x)-5f_1^{ij}(x))z^3\\ +(\epsilon(x)-f_3^{ij}(x)+5f_1^{ij}(x))z^4\end{array}}{(1-z)(1-5z^2+4z^4)}.$$

Hence the result. □

Now we can identify the ordinary generating functions of each sequence of the six sequences $f_n^{ij}(a), f_n^{ij}(b), f_n^{ij}(c), f_n^{ij}(\overline{a}), f_n^{ij}(\overline{b})$ and $f_n^{ij}(\overline{c})$.

Corollary 8. *The ordinary generating functions of the sequences that count the number of each elementary move in the set $\mathcal{A}$ are*

$$G^{ij}(z,a)=\frac{z-3z^3}{(1+z)(1-z)^2(1-4z^2)}, \tag{3.17}$$

$$G^{ij}(z,b)=\frac{z^3}{(1+z)(1-z)^2(1-4z^2)}, \tag{3.18}$$

$$G^{ij}(z,c)=\frac{z^3}{(1+z)(1-z)^2(1-4z^2)}, \tag{3.19}$$

$$G^{ij}(z,\overline{a})=\frac{2z^4}{(1+z)(1-z)^2(1-4z^2)}, \tag{3.20}$$

$$G^{ij}(z,\overline{b})=\frac{z^2-2z^4}{(1+z)(1-z)^2(1-4z^2)}, \tag{3.21}$$

$$G^{ij}(z,\overline{c})=\frac{z^2-2z^4}{(1+z)(1-z)^2(1-4z^2)}. \tag{3.22}$$

The following theorem presents the relation between $f_n^{ij}(x)$ and the sequence φ_n presented in the previous section.

Theorem 4. *For all integers $n \geq 0$, $x \in \mathcal{A}$ we have*

$$\begin{aligned}f_n^{ij}(x)=&[x\in\{a\}]\varphi_n+[x\in\{\overline{c},\overline{b}\}]\varphi_{n-1}+([x\in\{b,c\}]-3[x\in\{a\}])\varphi_{n-2}\\ &+(2[x\in\{\overline{a}\}]-2[x\in\{\overline{c},\overline{b}\}])\varphi_{n-3}.\end{aligned} \tag{3.23}$$

Proof. We have

$$
\begin{aligned}
G^{ij}(z,x) &= \frac{\begin{aligned}[x\in\{a\}]z+[x\in\{\overline{c},\overline{b}\}]z^2+([x\in\{b,c\}]-3[x\in\{a\}])z^3\\ +(2[x\in\{\overline{a}\}]-2[x\in\{\overline{c},\overline{b}\}])z^4\end{aligned}}{(1-4z^2)(1+z)(1-z)^2}\\
&= ([x\in\{a\}]+[x\in\{\overline{c},\overline{b}\}]z+([x\in\{b,c\}]-3[x\in\{a\}])z^2\\
&\quad +(2[x\in\{\overline{a}\}]-2[x\in\{\overline{c},\overline{b}\}])z^3)P(z)\\
&= \sum_{n\geq 0}([x\in\{a\}]\varphi_n+[x\in\{\overline{c},\overline{b}\}]\varphi_{n-1}+([x\in\{b,c\}]-3[x\in\{a\}])\varphi_{n-2}\\
&\quad +(2[x\in\{\overline{a}\}]-2[x\in\{\overline{c},\overline{b}\}])\varphi_{n-3})z^n. \qquad \square
\end{aligned}
$$

From this last theorem and by using Property 1, we find the following result.

Corollary 9. *For all integers $n\geq 0$, $x\in\mathcal{A}$, we have*

$$
f_n^{ij}(x)=\begin{cases}
\begin{aligned}&[x\in\{a,\overline{c},\overline{b}\}]\varphi_n+([x\in\{b,c\}]+2[x\in\{\overline{a}\}]-3[x\in\{a\}]\\ &-2[x\in\{b,c\}])\varphi_{n-2},\end{aligned} & \textit{if } n \textit{ even;}\\[2ex]
\begin{aligned}&[x\in\{a\}]\varphi_n+([x\in\{\overline{c},b,c,\overline{b}\}]-3[x\in\{a\}])\varphi_{n-1}\\ &+2([x\in\{\overline{a}\}]-[x\in\{\overline{c},\overline{b}\}])\varphi_{n-3},\end{aligned} & \textit{otherwise.}
\end{cases}
$$

By reordering the terms of the right side of equation (3.23) we find the identity

$$
\begin{aligned}
f_n^{ij}(x) &= (\varphi_n-3\varphi_{n-2})[x\in\{a\}]+\varphi_{n-2}[x\in\{b,c\}]+2\varphi_{n-3}[x\in\{\overline{a}\}]\\
&\quad +(\varphi_{n-1}-2\varphi_{n-3})[x\in\{\overline{b},\overline{c}\}].
\end{aligned} \tag{3.24}
$$

Replacing x with each possible value of the six values, we find the relation between the sequence of the number of each elementary move and φ_n.

Corollary 10. *For all integers $n\geq 0$, we have*

$$f_n^{ij}(a)=\varphi_n-3\varphi_{n-2}, \tag{3.25}$$
$$f_n^{ij}(b)=\varphi_{n-2}, \tag{3.26}$$
$$f_n^{ij}(c)=\varphi_{n-2}, \tag{3.27}$$
$$f_n^{ij}(\overline{a})=2\varphi_{n-3}, \tag{3.28}$$
$$f_n^{ij}(\overline{b})=\varphi_{n-1}-2\varphi_{n-3}, \tag{3.29}$$
$$f_n^{ij}(\overline{c})=\varphi_{n-1}-2\varphi_{n-3}. \tag{3.30}$$

Using Corollaries 3 and 10, we can find an explicit formula for each one of the six sequences $f_n^{ij}(a), f_n^{ij}(b), f_n^{ij}(c), f_n^{ij}(\overline{a}), f_n^{ij}(\overline{b})$ and $f_n^{ij}(\overline{c})$, as the following proposition presents.

Proposition 2. *For all integers $n \geq 0$ we have*

$$f_n^{ij}(a) = \frac{1}{9}(4^{\lfloor \frac{n+1}{2} \rfloor} + 6\left\lfloor \frac{n+1}{2} \right\rfloor - 1) = \begin{cases} \frac{1}{9}(2^n + 3n - 1), & \text{if } n \text{ even;} \\ \frac{1}{9}(2^{n+1} + 3n + 2), & \text{otherwise.} \end{cases} \tag{3.31}$$

$$f_n^{ij}(b) = \frac{1}{9}(4^{\lfloor \frac{n-1}{2} \rfloor + 1} - 3\left\lfloor \frac{n-1}{2} \right\rfloor - 4) = \begin{cases} \frac{1}{18}(2^{n+1} - 3n - 2), & \text{if } n \text{ even;} \\ \frac{1}{18}(2^{n+2} - 3n - 5), & \text{otherwise.} \end{cases} \tag{3.32}$$

$$f_n^{ij}(c) = \frac{1}{9}(4^{\lfloor \frac{n-1}{2} \rfloor + 1} - 3\left\lfloor \frac{n-1}{2} \right\rfloor - 4) = \begin{cases} \frac{1}{18}(2^{n+1} - 3n - 2), & \text{if } n \text{ even;} \\ \frac{1}{18}(2^{n+2} - 3n - 5), & \text{otherwise.} \end{cases} \tag{3.33}$$

$$f_n^{ij}(\overline{a}) = \frac{2}{9}(4^{\lfloor \frac{n-2}{2} \rfloor + 1} - 3\left\lfloor \frac{n-2}{2} \right\rfloor - 4) = \begin{cases} \frac{1}{9}(2^{n+1} - 3n - 2), & \text{if } n \text{ even;} \\ \frac{1}{9}(2^{n} - 3n + 1), & \text{otherwise.} \end{cases} \tag{3.34}$$

$$f_n^{ij}(\overline{b}) = \frac{1}{18}(4^{\lfloor \frac{n+2}{2} \rfloor} + 6\left\lfloor \frac{n+2}{2} \right\rfloor - 10) = \begin{cases} \frac{1}{18}(2^{n+2} + 3n - 4), & \text{if } n \text{ even;} \\ \frac{1}{18}(2^{n+1} + 3n - 7), & \text{otherwise.} \end{cases} \tag{3.35}$$

$$f_n^{ij}(\overline{c}) = \frac{1}{18}(4^{\lfloor \frac{n+2}{2} \rfloor} + 6\left\lfloor \frac{n+2}{2} \right\rfloor - 10) = \begin{cases} \frac{1}{18}(2^{n+2} + 3n - 4), & \text{if } n \text{ even;} \\ \frac{1}{18}(2^{n+1} + 3n - 7), & \text{otherwise.} \end{cases} \tag{3.36}$$

Depending on the parity of the number of discs n, we can now find which movement among the six elementary movements is the most frequent in the optimal sequence of movements, as the following proposition presents.

Proposition 3. *For all integers $n \geq 0$, we have*

$$\begin{aligned} &f_n^{ij}(\overline{c}) = f_n^{ij}(\overline{b}) \geq f_n^{ij}(\overline{a}) \geq f_n^{ij}(a) \geq f_n^{ij}(b) = f_n^{ij}(c), \quad \text{if } n \geq 6 \text{ even.} \\ &f_n^{ij}(a) \geq f_n^{ij}(b) = f_n^{ij}(c) \geq f_n^{ij}(\overline{c}) = f_n^{ij}(\overline{b}) \geq f_n^{ij}(\overline{a}), \quad \text{if } n \geq 0 \text{ odd.} \end{aligned}$$

This last proposition can be proved by induction on the number of discs n. We deduce from this proposition that the counter-clockwise moves $\overline{a}$, $\overline{b}$, and $\overline{c}$ are more than clockwise moves a, b, and c when n is even and vice versa when n is odd.

Now we present the relation between the number of the three clockwise (resp., counter-clockwise) moves a, b, and c (resp., $\overline{a}$, $\overline{b}$, and $\overline{c}$), as well as the relation between the number of moves from the source peg i to destination peg j which are moves of type a and the moves in the opposite direction which are moves of type $\overline{a}$.

Proposition 4. *For all integers $n \geq 0$, we have*

$$f_n^{ij}(a) = f_n^{ij}(b) + \left\lfloor \frac{n+1}{2} \right\rfloor = f_n^{ij}(c) + \left\lfloor \frac{n+1}{2} \right\rfloor,$$

$$f_n^{ij}(\overline{a}) = f_n^{ij}(\overline{c}) - \left\lfloor \frac{n}{2} \right\rfloor = f_n^{ij}(\overline{b}) - \left\lfloor \frac{n}{2} \right\rfloor,$$

$$f_n^{ij}(a) - f_n^{ij}(\overline{a}) = \frac{1}{9}((-1)^{n+1}2^n + 6n + 1).$$

Here is another recurrence relation of $f_n^{ij}(x)$ which relates $f_n^{ij}(x)$ with $f_{n-2}^{ij}(x)$.

Theorem 5. *For all integers $n \geq 2$, $x \in \mathcal{A}$, $f_n^{ij}(x)$ satisfies the following recurrence relation*

$$\begin{aligned} f_n^{ij}(x) &= 4f_{n-2}^{ij}(x) + (3 - 2\left\lfloor \frac{n+1}{2} \right\rfloor)[x \in \{a\}] + \left\lfloor \frac{n-1}{2} \right\rfloor [x \in \{b,c\}] \\ &\quad + 2\left\lfloor \frac{n-2}{2} \right\rfloor [x \in \{\overline{a}\}] + (2 - \left\lfloor \frac{n}{2} \right\rfloor)[x \in \{\overline{b},\overline{c}\}]. \end{aligned} \tag{3.37}$$

Proof. Using identities (3.24) and (2.1), we obtain

$$\begin{aligned} f_n^{ij}(x) &= (\varphi_n - 3\varphi_{n-2})[x \in \{a\}] + \varphi_{n-2}[x \in \{b,c\}] + 2\varphi_{n-3}[x \in \{\overline{a}\}] \\ &\quad + (\varphi_{n-1} - 2\varphi_{n-3})[x \in \{\overline{b},\overline{c}\}] \\ &= (4\varphi_{n-2} + \left\lfloor \frac{n+1}{2} \right\rfloor - 12\varphi_{n-4} - 3\left\lfloor \frac{n-1}{2} \right\rfloor)[x \in \{a\}] \\ &\quad + (4\varphi_{n-4} + \left\lfloor \frac{n-1}{2} \right\rfloor)[x \in \{b,c\}] + 2(4\varphi_{n-5} + \left\lfloor \frac{n-2}{2} \right\rfloor)[x \in \{\overline{a}\}] \\ &\quad + (4\varphi_{n-3} + \left\lfloor \frac{n}{2} \right\rfloor - 8\varphi_{n-5} - 2\left\lfloor \frac{n-2}{2} \right\rfloor)[x \in \{\overline{b},\overline{c}\}] \\ &= 4f_{n-2}^{ij}(x) + (\left\lfloor \frac{n+1}{2} \right\rfloor - 3\left\lfloor \frac{n-1}{2} \right\rfloor)[x \in \{a\}] + \left\lfloor \frac{n-1}{2} \right\rfloor [x \in \{b,c\}] \\ &\quad + 2\left\lfloor \frac{n-2}{2} \right\rfloor [x \in \{\overline{a}\}] + (\left\lfloor \frac{n}{2} \right\rfloor - 2\left\lfloor \frac{n-2}{2} \right\rfloor)[x \in \{\overline{b},\overline{c}\}] \\ &= 4f_{n-2}^{ij}(x) + (3 - 2\left\lfloor \frac{n+1}{2} \right\rfloor)[x \in \{a\}] + \left\lfloor \frac{n-1}{2} \right\rfloor [x \in \{b,c\}] + 2\left\lfloor \frac{n-2}{2} \right\rfloor [x \in \{\overline{a}\}] \\ &\quad + (2 - \left\lfloor \frac{n}{2} \right\rfloor)[x \in \{\overline{b},\overline{c}\}]. \end{aligned}$$

∎

Corollary 11. *For all integers $n \geq 2$, we have*

$$f_n^{ij}(a) = 4f_{n-2}^{ij}(a) + 3 - 2\left\lfloor \frac{n+1}{2} \right\rfloor, \quad f_0^{ij}(a) = 0,\ f_1^{ij}(a) = 1. \tag{3.38}$$

$$f_n^{ij}(b) = 4f_{n-2}^{ij}(b) + \left\lfloor \frac{n-1}{2} \right\rfloor, \quad f_0^{ij}(b) = f_1^{ij}(b) = 0. \tag{3.39}$$

$$f_n^{ij}(c) = 4f_{n-2}^{ij}(c) + \left\lfloor \frac{n-1}{2} \right\rfloor, \quad f_0^{ij}(c) = f_1^{ij}(c) = 0. \tag{3.40}$$

$$f_n^{ij}(\overline{a}) = 4f_{n-2}^{ij}(\overline{a}) + 2\left\lfloor \frac{n-2}{2} \right\rfloor, \quad f_0^{ij}(\overline{a}) = f_1^{ij}(\overline{a}) = 0. \tag{3.41}$$

$$f_n^{ij}(\overline{b}) = 4f_{n-2}^{ij}(\overline{b}) + 2 - \left\lfloor \frac{n}{2} \right\rfloor, \quad f_0^{ij}(\overline{b}) = f_1^{ij}(\overline{b}) = 0. \tag{3.42}$$

$$f_n^{ij}(\overline{c}) = 4f_{n-2}^{ij}(\overline{c}) + 2 - \left\lfloor \frac{n}{2} \right\rfloor, \quad f_0^{ij}(\overline{c}) = f_1^{ij}(\overline{c}) = 0. \tag{3.43}$$

Property 2. *For all integers $n \geq 1$, we have for*

$$f_{2n}^{ij}(a) = f_{2n-1}^{ij}(a),$$
$$f_{2n}^{ij}(b) = f_{2n-1}^{ij}(b),$$
$$f_{2n}^{ij}(c) = f_{2n-1}^{ij}(c).$$

And for all integers $n \geq 0$

$$f_{2n}^{ij}(\overline{a}) = f_{2n+1}^{ij}(\overline{a}),$$
$$f_{2n}^{ij}(\overline{b}) = f_{2n+1}^{ij}(\overline{b}),$$
$$f_{2n}^{ij}(\overline{c}) = f_{2n+1}^{ij}(\overline{c}).$$

We present now the relation between sequences $f_n^{ij}(a), f_n^{ij}(b), f_n^{ij}(c), f_n^{ij}(\overline{a}), f_n^{ij}(\overline{b}), f_n^{ij}(\overline{c})$ and some OEIS sequences.

Remark 2.

$$f_{2n}^{ij}(a) = f_{2n-1}^{ij}(a) = A073724(n), \quad n \geq 1.$$
$$f_{2n}^{ij}(b) = f_{2n-1}^{ij}(b) = A014825(n-1), \quad n \geq 2.$$
$$f_{2n}^{ij}(c) = f_{2n-1}^{ij}(c) = A014825(n-1), \quad n \geq 2.$$
$$f_{2n}^{ij}(\overline{a}) = f_{2n+1}^{ij}(\overline{a}) = A145766(n-1), \quad n \geq 1.$$
$$f_{2n}^{ij}(\overline{b}) = f_{2n+1}^{ij}(\overline{b}) = A160156(n-1), \quad n \geq 1.$$
$$f_{2n}^{ij}(\overline{c}) = f_{2n+1}^{ij}(\overline{c}) = A160156(n-1), \quad n \geq 1.$$

4 Counting the number of each elementary move that each disc makes

In this section, we extend our work to calculate not only the number of each elementary move in the optimal sequence of moves but also the number of times that each disc $d \in \mathcal{D} = \{1, \ldots, n\}$ makes the elementary move $x \in \mathcal{A}$ during the execution of the optimal solution.

We define the recursive function

$$g_n^{ij} : \begin{cases} \mathcal{D} \times \mathcal{A} \to \mathbb{N}; \\ (d, x) \mapsto g_n^{ij}(d, x), \end{cases}$$

where $d \in \mathcal{D} = \{1, \ldots, n\}$ is a disc, $x \in \mathcal{A} = \{a, b, c, \overline{a}, \overline{b}, \overline{c}\}$ is an elementary move, and $g_n^{ij}(d, x)$ counts the number of times that disc d makes the elementary move x in the sequence of optimal moves to solve the Tower of Hanoi, where the goal is to transfer a tower of n discs from peg i to peg j using peg k as an auxiliary peg.

Proposition 5. *For all integers $n \geq 0$, $d \in \mathcal{D}$, $x \in \mathcal{A}$, we have*

$$\sum_{x\in\mathcal{A}} g_n^{ij}(d,x) = 2^{n-d}. \tag{4.1}$$

$$\sum_{d\in\mathcal{D}} g_n^{ij}(d,x) = f_n^{ij}(x). \tag{4.2}$$

$$g_n^{ij}(n,x) = [x \in \{a\}]. \tag{4.3}$$

Consider the optimal sequence of moves for $n = 6$, see Table 1. The following table shows the number of times that each disc makes each one of the six elementary moves in the optimal sequence of moves above where $n = 6$.

d	$g_n^{ij}(d,a)$	$g_n^{ij}(d,b)$	$g_n^{ij}(d,c)$	$g_n^{ij}(d,\overline{a})$	$g_n^{ij}(d,\overline{b})$	$g_n^{ij}(d,\overline{c})$	$\sum_{x\in\mathcal{A}} g_n^{ij}(d,x)$
1	0	0	0	10	11	11	2^{6-1}
2	6	5	5	0	0	0	2^{6-2}
3	0	0	0	2	3	3	2^{6-3}
4	2	1	1	0	0	0	2^{6-4}
5	0	0	0	0	1	1	2^{6-5}
6	1	0	0	0	0	0	2^{6-6}
$\sum_{d\in\mathcal{D}} g_n^{ij}(d,x)$	$f_6^{ij}(a)$	$f_6^{ij}(b)$	$f_6^{ij}(c)$	$f_6^{ij}(\overline{a})$	$f_6^{ij}(\overline{b})$	$f_6^{ij}(\overline{c})$	$2^6 - 1$

Table 3: The number of each elementary move that each disc makes during the optimal solution for $n = 6$.

We present in the following theorem a recurrence relation of $g_n^{ij}(d,x)$ where the recursion is on the number of discs n.

Theorem 6. *For all integers $n \geq 4$, $d \in \mathcal{D}$, $x \in \mathcal{A}$, we have $g_n^{ij}(d,x)$ satisfies the following recurrence relation*

$$\begin{aligned} g_n^{ij}(d,x) = {} & 5g_{n-2}^{ij}(d,x) - 4g_{n-4}^{ij}(d,x) + [x \in \{a\}][d = n] + [x \in \{\overline{c},\overline{b}\}][d = n-1] \\ & + ([x \in \{b,c\}] - 3[x \in \{a\}])[d = n-2] - 2([x \in \{\overline{c},\overline{b}\}] \\ & - [x \in \{\overline{a}\}])[d = n-3], \end{aligned} \tag{4.4}$$

where $g_0^{ij}(d,x) = 0$, $g_1^{ij}(d,x) = [x \in \{a\}][d = 1]$, $g_2^{ij}(d,x) = [x \in \{a\}][d = 2] + [x \in \{\overline{c},\overline{b}\}][d = 1]$, and $g_3^{ij}(d,x) = [x \in \{a\}][d = 3] + [x \in \{\overline{c},\overline{b}\}][d = 2] + ([x \in \{b,c\}] + 2[x \in \{a\}])[d = 1]$.

Proof. We have

$$S_n^{ij} = S_{n-1}^{ik}[i \to j]S_{n-1}^{kj},$$

then we obtain

$$\begin{aligned} g_n^{ij}(d,x) &= g_{n-1}^{ik}(d,x) + [x \in \{a\}][d = n] + g_{n-1}^{kj}(d,x) \\ &= g_{n-2}^{ij}(d,x) + [x \in \{\overline{c}\}][d = n-1] + g_{n-2}^{jk}(d,x) + [x \in \{a\}][d = n] + g_{n-2}^{ki}(d,x) \\ &\quad + [x \in \{\overline{b}\}][d = n-1] + g_{n-2}^{ij}(d,x) \\ &= 2g_{n-2}^{ij}(d,x) + g_{n-1}^{ji}(d,x) + [x \in \{a\}][d = n] + ([x \in \{\overline{c},\overline{b}\}] - [x \in \{\overline{a}\}])[d = n-1]. \end{aligned}$$

Which implies that

$$g_{n-1}^{ji}(d,x) = g_n^{ij}(d,x) - 2g_{n-2}^{ij}(d,x) - [x \in \{a\}][d=n] - ([x \in \{\overline{c},\overline{b}\}] + [x \in \{\overline{a}\}])[d=n-1]. \tag{4.5}$$

Then, we have

$$g_n^{ji}(d,x) = g_{n+1}^{ij}(d,x) - 2g_{n-1}^{ij}(d,x) - [x \in \{a\}][d=n+1] - ([x \in \{\overline{c},\overline{b}\}] + [x \in \{\overline{a}\}])[d=n-1], \tag{4.6}$$

and

$$g_{n-2}^{ji}(d,x) = g_{n-1}^{ij}(d,x) - 2g_{n-3}^{ij}(d,x) - [x \in \{a\}][d=n-1] - ([x \in \{\overline{c},\overline{b}\}] + [x \in \{\overline{a}\}])[d=n-1]. \tag{4.7}$$

On the other hand we have

$$g_n^{ji}(d,x) = 2g_{n-2}^{ji}(d,x) + g_{n-1}^{ij}(d,x) + [x \in \{\overline{a}\}][d=n] + ([x \in \{b,c\}] - [x \in \{a\}])[d=n-1]. \tag{4.8}$$

By replacing (4.6) and (4.7) in (4.8), we obtain

$$\begin{aligned} g_{n+1}^{ij}(d,x) = {} & 5g_{n-1}^{ij}(d,x) - 4g_{n-3}^{ij}(d,x) + [x \in \{a\}][d=n+1] + [x \in \{\overline{c},\overline{b}\}][d=n] \\ & + ([x \in \{b,c\}] - 3[x \in \{a\}])[d=n-1] - 2([x \in \{\overline{c},\overline{b}\}] \\ & - [x \in \{\overline{a}\}])[d=n-2]. \end{aligned} \tag{4.9}$$

Hence, the result. □

Theorem 7. *For all $d \in \mathcal{D}$, and $x \in \mathcal{A}$, let $H^{ij}(z,d,x) = \sum_{n\geq 0} g_n^{ij}(d,x)z^n$ be the ordinary generating function of the sequence $g_n^{ij}(d,x)$, then we have*

$$H^{ij}(z,d,x) = \frac{\begin{array}{r}[x \in \{a\}]z^d + [x \in \{\overline{c},\overline{b}\}]z^{d+1} + ([x \in \{b,c\}] - 3[x \in \{a\}])z^{d+2} \\ + 2([x \in \{\overline{a}\}] - [x \in \{\overline{c},\overline{b}\}])z^{d+3}\end{array}}{(1-4z^2)(1-z^2)}. \tag{4.10}$$

Proof. Using the same technique as in the proof of Theorem 3. □

We present now the relation between $g_n^{ij}(d,x)$ and the sequence of the second section φ_n.

Theorem 8. *For all integers $n \geq 0$, $d \in \mathcal{D}$, $x \in \mathcal{A}$, we have*

$$\begin{aligned} g_n^{ij}(d,x) = {} & [x \in \{a\}]\varphi_{n-d+1} + ([x \in \{\overline{c},\overline{b}\}] - [x \in \{a\}])\varphi_{n-d} + ([x \in \{b,c\}] \\ & - 3[x \in \{a\}] - [x \in \{\overline{c},\overline{b}\}])\varphi_{n-d-1} + (2[x \in \{\overline{a}\}] - 2[x \in \{\overline{c},\overline{b}\}] \\ & - [x \in \{b,c\}] + 3[x \in \{a\}])\varphi_{n-d-2} - 2([x \in \{\overline{a}\}] - [x \in \{\overline{c},\overline{b}\}])\varphi_{n-d-3}. \end{aligned} \tag{4.11}$$

Proof. Using the generating function $H^{ij}(t,d,x)$ and some elementary calculations. $\square$

Here we have another recurrence relation of $g_n^{ij}(d,x)$ where the recursion is on d, the disc whose moves we wish to count.

Proposition 6. *For all integers $n \geq 1$, $d \in \mathcal{D} \setminus \{1\}$, $x \in \mathcal{A}$ we have*

$$g_n^{ij}(d,x) = g_{n-1}^{ij}(d-1,x), \tag{4.12}$$

where $g_n^{ij}(1,x)$ is calculated using identity (4.11).

Proof. The proof comes from the fact that the label of a disc d is related to the value of the number of discs n in the tower to be transferred. When the size of the tower to be transferred is reduced by one, the label of disc d is also reduced by one, by convention that if the label becomes less than 1 we eliminate the disc. Then calculating $g_n^{ij}(d,x)$ which is the number of times disc d makes move x in a tower of n discs in the optimal sequence of moves that transfers the tower from peg i to j, the same as calculating $g_{n-1}^{ij}(d-1,x)$ because disc $d-1$ in the sub-tower of the last $n-1$ acts the same as disc d in the tower n.

The proposition can also be proved by induction on the value of d. $\square$

This last result allows us to identify triangles for the sequence $g_n^{ij}(d,x)$ where the move x is fixed, we present here an example.

$n \backslash d$	1	2	3	4	5	6	7	8	9	10	$\cdots$
1	1										
2	0	1									
3	2	0	1								
4	0	2	0	1							
5	6	0	2	0	1						
6	0	6	0	2	0	1					
7	22	0	6	0	2	0	1				
8	0	22	0	6	0	2	0	1			
9	86	0	22	0	6	0	2	0	1		
10	0	86	0	22	0	6	0	2	0	1	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Table 4: The triangle of $g_n^{ij}(d,a)$.

Proposition 7. *For all integers $n \geq 0$, $d \in \mathcal{D}$, we have*

$$g_n^{ij}(d,a) = g_n^{ij}(d,b) = g_n^{ij}(d,c) = 0, \quad (n-d) \text{ even.}$$
$$g_n^{ij}(d,\overline{a}) = g_n^{ij}(d,\overline{b}) = g_n^{ij}(d,\overline{c}) = 0, \quad (n-d) \text{ odd.}$$

Proof. Using identity (4.11) and Property 1. $\square$

The interpretation of this proposition is that the clockwise (resp., counter-clockwise) moves a, b, and c (resp., $\overline{a}$, $\overline{b}$, and $\overline{c}$) are null when the difference between n and d is an even (resp., odd) number.

Similarly to Proposition 1, we find that in the optimal solution, the number of times disc d is moved from the destination (resp., source) peg j (resp., i) to the auxiliary peg k is equal to the number of times we move discs from the auxiliary peg k to the source (resp., destination) peg i (resp., j). i.e.,

Proposition 8. *for all integers $n \geq 0$, $d \in \mathcal{D}$ we have*

$$g_n^{ij}(d,b) = g_n^{ij}(d,c),$$
$$g_n^{ij}(d,\overline{b}) = g_n^{ij}(d,\overline{c}).$$

Proof. By replacing x with b, c, $\overline{b}$, and $\overline{c}$ in identity (4.11). ∎

As a direct consequence of Theorem 8 we have the following result which presents the relation between the number of times disc d makes move $x \in \mathcal{A}$ in the optimal solution i.e., the six sequences $g_n^{ij}(d,a)$, $g_n^{ij}(d,b)$, $g_n^{ij}(d,c)$, $g_n^{ij}(d,\overline{a})$, $g_n^{ij}(d,\overline{b})$, and $g_n^{ij}(d,\overline{c})$ with the sequence φ_n.

Corollary 12. *For all integers $n \geq 0$, $d \in \mathcal{D}$ we have*

$$g_n^{ij}(d,a) = \varphi_{n-d+1} - \varphi_{n-d} - 3\varphi_{n-d-1} + 3\varphi_{n-d-2}, \tag{4.13}$$
$$g_n^{ij}(d,b) = g_n^{ij}(d,c) = \varphi_{n-d-1} - \varphi_{n-d-2}, \tag{4.14}$$
$$g_n^{ij}(d,\overline{a}) = 2\varphi_{n-d-2} - 2\varphi_{n-d-3}, \tag{4.15}$$
$$g_n^{ij}(d,\overline{b}) = g_n^{ij}(d,\overline{c}) = \varphi_{n-d} - \varphi_{n-d-1} - 2\varphi_{n-d-2} + 2\varphi_{n-d-3}. \tag{4.16}$$

We present now explicit formulas for each of the sequences $g_n^{ij}(d,a)$, $g_n^{ij}(d,b)$, $g_n^{ij}(d,c)$, $g_n^{ij}(d,\overline{a})$, $g_n^{ij}(d,\overline{b})$, and $g_n^{ij}(d,\overline{c})$.

Corollary 13. *For all integers $n \geq 0$, $d \in \mathcal{D}$, we have*

$$g_n^{ij}(d,a) = \begin{cases} \frac{1}{3}(2^{n-d}+2), & (n-d) \text{ even;} \\ 0, & \text{otherwise.} \end{cases} \tag{4.17}$$

$$g_n^{ij}(d,b) = g_n^{ij}(d,c) = \begin{cases} \frac{1}{3}(2^{n-d}-1), & (n-d) \text{ even;} \\ 0, & \text{otherwise.} \end{cases} \tag{4.18}$$

$$g_n^{ij}(d,\overline{a}) = \begin{cases} 0, & (n-d) \text{ even;} \\ \frac{1}{3}(2^{n-d}-2), & \text{otherwise.} \end{cases} \tag{4.19}$$

$$g_n^{ij}(d,\overline{b}) = g_n^{ij}(d,\overline{c}) = \begin{cases} 0, & (n-d) \text{ even;} \\ \frac{1}{3}(2^{n-d}+1), & \text{otherwise.} \end{cases} \tag{4.20}$$

Proof. Using Corollaries 12 and 3, and Proposition 7. ∎

We finish this section by presenting the relationship between sequences $g_n^{ij}(d,a)$, $g_n^{ij}(d,b)$, $g_n^{ij}(d,c)$, $g_n^{ij}(d,\overline{a})$, $g_n^{ij}(d,\overline{b})$, and $g_n^{ij}(d,\overline{c})$ with some OEIS sequences, due to Proposition 6 we can fix $d=1$.

Remark 3. *For all integers $n \geq 0$, we have*

$$g_{2n+1}^{ij}(1,a) = A047849(n),$$

$$g_{2n+1}^{ij}(1,b) = A002450(n),$$

$$g_{2n+1}^{ij}(1,c) = A002450(n),$$

$$g_{2n+2}^{ij}(1,\overline{a}) = A020988(n),$$

$$g_{2n+2}^{ij}(1,\overline{b}) = A007583(n),$$

$$g_{2n+2}^{ij}(1,\overline{c}) = A007583(n).$$

5 Conclusion

The Tower of Hanoi puzzle still fascinates us by its richness of mathematical properties. However, there is a lot to be discovered around this puzzle. The goal of this work was to find a way to calculate the number of each elementary move in the optimal sequence of moves, as well as find the number of times a specified disc makes each elementary move. These goals were achieved by developing the functions $f_n^{ij}(x)$ and $g_n^{ij}(d,x)$ using the properties of the sequence φ_n. Many properties and combinatorial identities relating φ_n with $f_n^{ij}(x)$ and $g_n^{ij}(d,x)$ were presented. We also found combinatorial interpretations for some sequences that do not have a combinatorial interpretation yet in the On-Line Encyclopedia of Integer Sequences [14] using $f_n^{ij}(x)$ and $g_n^{ij}(d,x)$.

Acknowledgement *We are very grateful to the anonymous referee for the careful reading of our manuscript, and for the valuable comments and suggestions which have helped to improve the quality of this work.*

References

[1] J.-P. Allouche, D. Astoorian, J. Randall, J. Shallit, Morphisms, squarefree strings, and the Tower of Hanoi puzzle, *The American Mathematical Monthly* **101 (7)** (1994), 651–658.

[2] J.-P. Allouche, J. Shallit, *Automatic sequences: theory, applications, generalizations*, Cambridge University Press, Cambridge (2003).

[3] D. Arett, S. Dorée, Coloring and counting on the Tower of Hanoi graphs, *Mathematics Magazine* **83 (3)** (2010), 200–209.

[4] I. Gorgol, A. Lechowska, Anti-Ramsey number of Hanoi graphs, *Discussiones Mathematicae: Graph Theory* **39 (1)** (2019), 285–296.

[5] A. M. Hinz, Pascal's Triangle and the Tower of Hanoi, *The American Mathematical Monthly* **99 (6)** (1992), 538–544.

[6] A. M. Hinz, The Lichtenberg sequence, *Fibonacci Quarterly* **55 (1)** (2017), 2–12.

[7] A. M. Hinz, S. Klavžar, U. Milutinović, D. Parisse, C. Petr, Metric properties of the Tower of Hanoi graphs and Stern's diatomic sequence, *European Journal of Combinatorics* **26 (5)** (2005), 693–708.

[8] A. M. Hinz, S. Klavžar, C. Petr, *The Tower of Hanoi - Myths and Maths*, Second Edition, Springer/Birkhäuser, Cham (2018).

[9] A. M. Hinz, A. Schief, The average distance on the Sierpiński gasket, *Probability Theory and Related Fields* **87 (1)** (1990), 129–138.

[10] A. M. Hinz, P. K. Stockmeyer, Discovering Fibonacci numbers, Fibonacci words, and a Fibonacci fractal in the Tower of Hanoi, *Fibonacci Quart.* **57 (5)** (2019), 72–83.

[11] S. Klavžar, U. Milutinović, C. Petr, Combinatorics of topmost discs of multi-peg Tower of Hanoi problem, *Ars Combinatoria* **59** (2001), 55–64.

[12] S. Klavžar, U. Milutinović, C. Petr, Hanoi graphs and some classical numbers, *Expositiones Mathematicae* **23 (4)** (2005), 371–378.

[13] É. Lucas, *Récréations mathématiques, volume III*, Gauthier-Villars et fils (1883).

[14] OEIS Foundation Inc., The On-Line Encyclopedia of Integer Sequences, published electronically at http://oeis.org (2022).

Received: 23.05.2023
Revised: 07.08.2023
Accepted: 08.08.2023

(1) Faculty of Mathematics, University of Science and Technology Houari Boumediene,
RECITS Laboratory, BP 32, El Alia 16111, Algiers, Algeria
E-mail: `hacenebelbachir@gmail.com`

(2) Faculty of Mathematics, University of Science and Technology Houari Boumediene,
RECITS Laboratory, BP 32, El Alia 16111, Algiers, Algeria
E-mail: `mehiri314@gmail.com`