Bull. Math. Soc. Sci. Math. Roumanie
Tome 68 (116), No. 2, 2025, 173–184

On the nonexistence of harmonic and bi-harmonic maps
by
Ahmed Mohammed Cherif

Abstract

In this paper, we study the existence of harmonic and bi-harmonic maps into Riemannian manifolds admitting a conformal vector field, or a nontrivial Ricci solitons.

Key Words: Harmonic maps, bi-harmonic maps, Ricci solitons, conformal vector fields.
2020 Mathematics Subject Classification: 53C25, 53C43, 58E20.

1 Preliminaries and notations

We give some definitions. (1) Let (M, g) be a Riemannian manifold. By R^M and Ric^M we denote respectively the Riemannian curvature tensor and the Ricci curvature of (M, g). Thus R^M and Ric^M are defined by

$$R^M(X,Y)Z = \nabla^M_X \nabla^M_Y Z - \nabla^M_Y \nabla^M_X Z - \nabla^M_{[X,Y]} Z, \tag{1.1}$$

$$\operatorname{Ric}^M(X,Y) = \sum_{i=1}^{m} g(R^M(X,e_i)e_i, Y), \tag{1.2}$$

where ∇^M is the Levi-Civita connection with respect to g, $\{e_i\}_{i=1}^m$ is a local orthonormal frame field on (M, g), and $X, Y, Z \in \Gamma(TM)$. The divergence of $(0, p)$-tensor α on (M, g) is defined by

$$(\operatorname{div}^M \alpha)(X_1, ..., X_{p-1}) = \sum_{i=1}^{m} (\nabla_{e_i} \alpha)(e_i, X_1, ..., X_{p-1}), \tag{1.3}$$

where $X_1, ..., X_{p-1} \in \Gamma(TM)$, and $\nabla_{e_i}\alpha$ is the covariant derivative of α relative to e_i. Given a smooth function f on M, the gradient of f is defined by

$$g(\operatorname{grad}^M f, X) = X(f), \tag{1.4}$$

the Hessian of f is defined by

$$(\operatorname{Hess}^M f)(X,Y) = g(\nabla^M_X \operatorname{grad}^M f, Y), \tag{1.5}$$

where $X, Y \in \Gamma(TM)$ (for more details, see for example [14]).
(2) A vector field ξ on a Riemannian manifold (M, g) is called a conformal if $\mathcal{L}_\xi g = 2fg$, for some smooth function f on M, where $\mathcal{L}_\xi g$ is the Lie derivative of the metric g with respect to ξ, that is

$$g(\nabla^M_X \xi, Y) + g(\nabla^M_Y \xi, X) = 2fg(X,Y), \quad \forall X, Y \in \Gamma(TM). \tag{1.6}$$

The function $f = (\operatorname{div}^M \xi)/m$ is then called the potential function of the conformal vector field ξ. If ξ is conformal with constant potential function f, then it is called homothetic, while $f = 0$ it is Killing (see [1], [11], [18]).
(3) A Ricci soliton structure on a Riemannian manifold (M, g) is the choice of a smooth vector field ξ satisfying the soliton equation

$$\operatorname{Ric}^M + \frac{1}{2}\mathcal{L}_\xi g = \lambda g, \tag{1.7}$$

for some constant $\lambda \in \mathbb{R}$. The Ricci soliton (M, g, ξ, λ) is said to be shrinking, steady or expansive according to whether the coefficient λ appearing in equation (1.7) satisfies $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$. In the special case where $\xi = \operatorname{grad}^M f$, for some smooth function f on M, we say that $(M, g, \operatorname{grad}^M f, \lambda)$ is a gradient Ricci soliton with potential f. In this situation, the soliton equation reads

$$\operatorname{Ric}^M + \operatorname{Hess}^M f = \lambda g, \tag{1.8}$$

(see [8], [9], [16]). If $\xi = 0$, we recover the definition of an Einstein metric with Einstein constant λ. If (M, g) is not Einstein, we call the soliton nontrivial.
(4) A vector field ξ on a Riemannian manifold (M, g) is said to be a Jacobi-type vector field if it satisfies

$$\nabla^M_X \nabla^M_X \xi - \nabla^M_{\nabla^M_X X}\xi + R^M(\xi, X)X = 0, \quad \forall X \in \Gamma(TM). \tag{1.9}$$

Note that, there are Jacobi-type vector fields on a Riemannian manifold which are not Killing vector fields (see [5]). Note that, a homothetic vector field on a Riemannian manifold is a Jacobi-type vector field (see [13]).
(5) Let $\varphi : (M, g) \longrightarrow (N, h)$ be a smooth map between two Riemannian manifolds, $\tau(\varphi)$ the tension field of φ given by

$$\tau(\varphi) = \operatorname{trace}_g \nabla d\varphi = \sum_{i=1}^m \Big[\nabla^\varphi_{e_i} d\varphi(e_i) - d\varphi(\nabla^M_{e_i} e_i)\Big], \tag{1.10}$$

where ∇^M is the Levi-Civita connection of (M, g), ∇^φ denote the pull-back connection on $\varphi^{-1}TN$ and $\{e_i\}_{i=1}^m$ is a local orthonormal frame field on (M, g). Then φ is called harmonic if the tension field vanishes, i.e., $\tau(\varphi) = 0$ (see [1], [3], [7], [17]). We define the index form for harmonic maps by (see [4], [15])

$$I(v, w) = \int_M h(J_\varphi(v), w)v^g, \quad \forall v, w \in \Gamma(\varphi^{-1}TN), \tag{1.11}$$

(or over any compact subset $D \subset M$), where

$$\begin{aligned} J_\varphi(v) &= -\operatorname{trace}_g R^N(v, d\varphi)d\varphi - \operatorname{trace}_g(\nabla^\varphi)^2 v \\ &= -\sum_{i=1}^m R^N(v, d\varphi(e_i))d\varphi(e_i) - \sum_{i=1}^m \Big[\nabla^\varphi_{e_i}\nabla^\varphi_{e_i} v - \nabla^\varphi_{\nabla^M_{e_i} e_i} v\Big], \end{aligned} \tag{1.12}$$

R^N is the curvature tensor of (N, h), ∇^N is the Levi-Civita connection of (N, h), and v^g is the volume form of (M, g) (see [1]). If $\tau_2(\varphi) \equiv J_\varphi(\tau(\varphi))$ is null on M, then φ is called a bi-harmonic map (see [3], [10], [12]).

2 Main results

2.1 Harmonic maps and conformal vector fields

Proposition 1. *Let (M, g) be a compact orientable Riemannian manifold without boundary, and (N, h) a Riemannian manifold admitting a conformal vector field ξ with potential function $f > 0$ at any point. Then, any harmonic map φ from (M, g) to (N, h) is constant.*

Proof. Let $X \in \Gamma(TM)$, we set

$$\omega(X) = h\big(\xi \circ \varphi, d\varphi(X)\big). \tag{2.1}$$

Let $\{e_i\}_{i=1}^m$ be a geodesic frame field around $x \in M$. At x we have

$$\operatorname{div}^M \omega = \sum_{i=1}^m e_i\big[h\big(\xi \circ \varphi, d\varphi(e_i)\big)\big], \tag{2.2}$$

by equation (2.2), and the harmonicity condition of φ, we get

$$\operatorname{div}^M \omega \quad = \quad \sum_{i=1}^m h\big(\nabla^\varphi_{e_i}(\xi \circ \varphi), d\varphi(e_i)\big), \tag{2.3}$$

since ξ is a conformal vector field, we find that

$$\operatorname{div}^M \omega = (f \circ \varphi) \sum_{i=1}^m h\big(d\varphi(e_i), d\varphi(e_i)\big) = (f \circ \varphi)|d\varphi|^2. \tag{2.4}$$

The Proposition 1 follows from equation (2.4), and the divergence theorem (see [1]), with $f > 0$ on N. ∎

Remark 1. *(1) Proposition 1 remains true if the potential function $f < 0$ on N (consider the conformal vector field $\bar{\xi} = -\xi$).*
(2) If the potential function is non-zero constant, that is $\mathcal{L}_\xi h = 2kh$ on (N, h) with $k \neq 0$, then any harmonic map φ from a compact orientable Riemannian manifold without boundary (M, g) to (N, h) is necessarily constant (see [13]).
(3) An harmonic map from a compact orientable Riemannian manifold without boundary to a Riemannian manifold admitting a Killing vector field is not necessarily constant (for example the identity map on the unit $(2n+1)$-dimensional sphere on $\mathbb{R}^{2n+2}$, note that the unit odd-dimensional sphere admits a Killing vector field (see [2]).

From Proposition 1, we get the following result.

Corollary 1. *Let $(\overline{N}, \overline{h})$ be an n-dimensional Riemannian manifold which admits a Killing vector field $\overline{\xi}$. Consider (N, h) a Riemannian hypersurface of $(\overline{N}, \overline{h})$ such that h is the induced metric of $\overline{h}$ on N. Suppose that*

- *(N,h) is totally umbilical, that is*

$$B(X,Y)=\rho h(X,Y)\eta,\quad \forall X,Y\in\Gamma(TN),$$

for some smooth function ρ on N, where B is the second fundamental form of N on $\overline{N}$ given by $B(X,Y)=(\overline{\nabla}_X Y)^{\perp}$, $\overline{\nabla}$ is the Levi-Civita connection on $\overline{N}$, and η is the unit normal to N;

- *the function $\overline{h}(\overline{\xi},H)\neq 0$ everywhere on N, where H is the mean curvature of (N,h) given by the formula*

$$H=\frac{1}{n-1}\operatorname{trace}_h B.$$

Then, any harmonic map from a compact orientable Riemannian manifold without boundary to (N,h) is constant.

Proof. It is possible to express $\overline{\xi}$ along N as $\overline{\xi}=\xi+f\eta$, where ξ is tangent to N and f is a smooth function on N. Thus we have

$$(\mathcal{L}_{\overline{\xi}}\overline{h})(X,Y)=(\mathcal{L}_{\xi}h)(X,Y)+f\{\overline{h}(\overline{\nabla}_X\eta,Y)+\overline{h}(\overline{\nabla}_Y\eta,X)\},\tag{2.5}$$

where $X,Y\in\Gamma(TN)$ (see [6]), by equation (2.5) with $\mathcal{L}_{\overline{\xi}}\overline{h}=0$, we get

$$(\mathcal{L}_{\xi}h)(X,Y)=2f\overline{h}(\eta,B(X,Y)).\tag{2.6}$$

Since (N,h) is totally umbilical, (2.6) becomes

$$(\mathcal{L}_{\xi}h)(X,Y)=2f\rho h(X,Y).\tag{2.7}$$

The Corollary follows from Proposition 1 and equation (2.7) with

$$f\rho=\overline{h}(\overline{\xi},\eta)\overline{h}(H,\eta)=\overline{h}(\overline{\xi},H).$$

∎

Example 1. *Let $\overline{N}=\mathbb{R}^n$ equipped with the standard Riemannian metric $\overline{h}=\langle,\rangle_{\mathbb{R}^n}$. We consider the hypersurface $N=\mathbb{S}^{n-1}\cap\{y\in\mathbb{R}^n, y_n>0\}$ in $\overline{N}$, where $\mathbb{S}^{n-1}$ is the unit $(n-1)$-dimensional sphere on $\mathbb{R}^n$. Let h the induced Riemannian metric on N. It is easy to show that (N,h) is totally umbilical, with $B(X,Y)=-h(X,Y)P$ for all $X,Y\in\Gamma(TN)$, where P is the position vector field of $\mathbb{R}^n$ (see [14]). Here, $\rho=1$ and $\eta=-P$ along N. We have the following*

$$\overline{h}(\frac{\partial}{\partial y_n},H)=-y_n\neq 0,$$

everywhere on N. Since $\overline{\xi}=\partial/\partial y_n$ is a Killing vector field on $(\overline{N},\overline{h})$, according to Corollary 1, any harmonic map from a compact orientable Riemannian manifold without boundary to an open hemisphere is constant. Here, the tangent vector field ξ is given by

$$\begin{aligned}\xi &= \overline{\xi}-f\eta\\ &= \frac{\partial}{\partial y_n}-\langle\frac{\partial}{\partial y_n},P\rangle_{\mathbb{R}^n}P\\ &= \frac{\partial}{\partial y_n}-y_n\sum_{i=1}^{n}y_i\frac{\partial}{\partial y_i}.\end{aligned}$$

Note that, the hypersurface (N,h) can be parameterized by

$$(u_1,...,u_{n-1}) \longmapsto \Big(u_1,...,u_{n-1},\sqrt{1-u_1^2-...-u_{n-1}^2}\Big).$$

By using this parameterization and the last equation, we find that

$$\xi = -\sqrt{1-u_1^2-...-u_{n-1}^2}\sum_{i=1}^{n-1} u_i \frac{\partial}{\partial u_i}.$$

Thus, from equation (2.7), ξ is conformal vector field on (N,h) with potential function

$$f = -\sqrt{1-u_1^2-...-u_{n-1}^2}.$$

In the case of non-compact Riemannian manifold, we obtain the following results.

Theorem 1. *Let (M,g) be a complete non-compact Riemannian manifold, and (N,h) a Riemannian manifold admitting a conformal vector field ξ with potential function $f>0$ at any point. If $\varphi:(M,g)\longrightarrow(N,h)$ is harmonic map, satisfying*

$$\int_M \frac{|\xi\circ\varphi|^2}{f\circ\varphi} v^g < \infty, \tag{2.8}$$

then φ is constant.

Proof. Let ρ be a smooth function with compact support on M, we set

$$\omega(X) = h\big(\xi\circ\varphi, \rho^2 d\varphi(X)\big), \quad X\in\Gamma(TM). \tag{2.9}$$

Let $\{e_i\}_{i=1}^m$ be a geodesic frame field around $x\in M$. At x we have

$$\operatorname{div}^M \omega = \sum_{i=1}^m e_i\big[h\big(\xi\circ\varphi,\rho^2 d\varphi(e_i)\big)\big], \tag{2.10}$$

by equation (2.10), and the harmonicity condition of φ, we get

$$\begin{aligned}\operatorname{div}^M \omega &= \sum_{i=1}^m \Big[h\big(\nabla^\varphi_{e_i}(\xi\circ\varphi),\rho^2 d\varphi(e_i)\big) + h\big(\xi\circ\varphi,\nabla^\varphi_{e_i}\rho^2 d\varphi(e_i)\big)\Big]\\ &= \sum_{i=1}^m \Big[\rho^2 h\big(\nabla^\varphi_{e_i}(\xi\circ\varphi), d\varphi(e_i)\big) + 2\rho e_i(\rho) h\big(\xi\circ\varphi, d\varphi(e_i)\big)\Big],\end{aligned} \tag{2.11}$$

since ξ is a conformal vector field with potential function f, we find that

$$\rho^2\sum_{i=1}^m h\big(\nabla^\varphi_{e_i}(\xi\circ\varphi), d\varphi(e_i)\big) = (f\circ\varphi)\rho^2\sum_{i=1}^m h\big(d\varphi(e_i), d\varphi(e_i)\big), \tag{2.12}$$

by Young's inequality we have

$$-2\rho\sum_{i=1}^m e_i(\rho) h\big(\xi\circ\varphi, d\varphi(e_i)\big) \le \lambda\rho^2|d\varphi|^2 + \frac{1}{\lambda}|\operatorname{grad}^M\rho|^2|\xi\circ\varphi|^2, \tag{2.13}$$

for all function $\lambda > 0$ on M, because of the inequality

$$\sum_{i=1}^{m} |\sqrt{\lambda}\rho d\varphi(e_i) + \frac{1}{\sqrt{\lambda}} e_i(\rho)(\xi \circ \varphi)|^2 \geq 0.$$

From (2.11), (2.12) and (2.13) we deduce the inequality

$$(f \circ \varphi)\rho^2 |d\varphi|^2 - \operatorname{div}^M \omega \leq \lambda \rho^2 |d\varphi|^2 + \frac{1}{\lambda} |\operatorname{grad}^M \rho|^2 |\xi \circ \varphi|^2. \tag{2.14}$$

Take $\lambda = (f \circ \varphi)/2$, from (2.14) we have

$$\frac{1}{2}(f \circ \varphi)\rho^2 |d\varphi|^2 - \operatorname{div}^M \omega \leq \frac{2}{f \circ \varphi} |\operatorname{grad}^M \rho|^2 |\xi \circ \varphi|^2, \tag{2.15}$$

by using the divergence theorem and inequality (2.15), we find that

$$\frac{1}{2} \int_M (f \circ \varphi)\rho^2 |d\varphi|^2 v^g \leq 2 \int_M |\operatorname{grad}^M \rho|^2 \frac{|\xi \circ \varphi|^2}{f \circ \varphi} v^g. \tag{2.16}$$

Consider the smooth function $\rho = \rho_R$ such that, $\rho \leq 1$ on M, $\rho = 1$ on the ball $B(p, R)$, $\rho = 0$ on $M \backslash B(p, 2R)$ and $|\operatorname{grad}^M \rho| \leq \frac{2}{R}$ (see [19]). From (2.16), we get

$$\frac{1}{2} \int_M (f \circ \varphi)\rho^2 |d\varphi|^2 v^g \leq \frac{8}{R^2} \int_M \frac{|\xi \circ \varphi|^2}{f \circ \varphi} v^g, \tag{2.17}$$

since $\int_M \frac{|\xi \circ \varphi|^2}{f \circ \varphi} v^g < \infty$, when $R \to \infty$, we obtain

$$\int_M (f \circ \varphi)|d\varphi|^2 v^g = 0. \tag{2.18}$$

Consequently, $|d\varphi| = 0$, that is φ is constant. ∎

From Theorem 1, we get the following.

Corollary 2. *Let (M, g) be a complete non-compact Riemannian manifold and let ξ a conformal vector field on (M, g) with potential function $f > 0$ at any point. Then*

$$\int_M \frac{|\xi|^2}{f} v^g = \infty.$$

2.2 Bi-harmonic maps and conformal vector fields

Theorem 2. *Let (M, g) be a compact orientable Riemannian manifold without boundary, and let ξ a conformal vector field with non-constant potential function f on a Riemannian manifold (N, h) such that $\operatorname{grad}^N f$ is parallel. Then, any bi-harmonic map φ from (M, g) to (N, h) is constant.*

For the proof of Theorem 2, we need the following Lemma.

Lemma 1. *[13] Let (M, g) be a compact orientable Riemannian manifold without boundary and (N, h) a Riemannian manifold admitting a proper homothetic vector field ζ, i.e., $\mathcal{L}_\zeta h = 2kh$ with $k \in \mathbb{R}^*$. Then, any bi-harmonic map φ from (M, g) to (N, h) is constant.*

Proof of Theorem 2. We set $\zeta = [\operatorname{grad}^N f, \xi]$, since $\operatorname{grad}^N f$ is parallel on (N, h), then ζ is an homothetic vector field satisfying $\nabla^N_U \zeta = |\operatorname{grad}^N f|^2 U$ for any $U \in \Gamma(TN)$ (see [11]). The Theorem 2 follows from Lemma 1. □

From Theorem 2, we deduce:

Corollary 3. *Let (M, g) be a compact orientable Riemannian manifold without boundary, and let ξ a conformal vector field with non-constant potential function f on (M, g). Then,* $\operatorname{grad} f$ *is not parallel.*

2.3 Harmonic maps to Ricci solitons

Proposition 2. *Let (M, g) be a compact orientable Riemannian manifold without boundary, and (N, h, ξ, λ) a nontrivial Ricci soliton with*

$$\operatorname{Ric}^N > \lambda h \quad \text{or} \quad \operatorname{Ric}^N < \lambda h.$$

Then any harmonic map φ from (M, g) to (N, h) is constant.

Proof. Let $X \in \Gamma(TM)$, we set

$$\omega(X) = h\big(\xi \circ \varphi, d\varphi(X)\big). \tag{2.19}$$

Let $\{e_i\}_{i=1}^m$ be a geodesic frame field around $x \in M$. At x we have

$$\operatorname{div}^M \omega = \sum_{i=1}^m e_i\big[h\big(\xi \circ \varphi, d\varphi(e_i)\big)\big], \tag{2.20}$$

by equation (2.20), and the harmonicity condition of φ, we get

$$\operatorname{div}^M \omega = \sum_{i=1}^m h\big(\nabla^\varphi_{e_i}(\xi \circ \varphi), d\varphi(e_i)\big) = \frac{1}{2}\sum_{i=1}^m (\mathcal{L}_\xi h)(d\varphi(e_i), d\varphi(e_i)), \tag{2.21}$$

from the soliton equation, we find that

$$\operatorname{div}^M \omega = \lambda \sum_{i=1}^m h\big(d\varphi(e_i), d\varphi(e_i)\big) - \sum_{i=1}^m \operatorname{Ric}^N(d\varphi(e_i), d\varphi(e_i)). \tag{2.22}$$

The Proposition 2 follows from equation (2.22), and the divergence theorem. □

Remark 2. *The condition* $\operatorname{Ric}^N > \lambda h$ *(resp.* $\operatorname{Ric}^N < \lambda h$*) is equivalent to* $\operatorname{Ric}^N(v, v) > \lambda h(v, v)$ *(resp.* $\operatorname{Ric}^N(v, v) < \lambda h(v, v)$*), for any $v \in T_pN - \{0\}$, where $p \in N$.*

It is known that the cigar soliton

$$(\mathbb{R}^2, \frac{dx^2+dy^2}{1+x^2+y^2}),$$

is steady with strictly positive Ricci tensor (see [8]). According to Proposition 2, we have the following

Corollary 4. *Any harmonic map φ from a compact orientable Riemannian manifold without boundary to the cigar soliton is constant.*

In the case of non-compact Riemannian manifold, we obtain the following results.

Theorem 3. *Let (M,g) be a complete non-compact Riemannian manifold, and (N,h,ξ,λ) a nontrivial Ricci soliton with $\operatorname{Ric}^N < \mu h$, for some constant $\mu < \lambda$. If $\varphi : (M,g) \longrightarrow (N,h)$ is harmonic map, satisfying*

$$\int_M |\xi\circ\varphi|^2 v^g < \infty, \tag{2.23}$$

then φ is constant.

Proof. Let ρ be a smooth function with compact support on M, we set

$$\omega(X) = h\big(\xi\circ\varphi, \rho^2 d\varphi(X)\big), \quad X\in\Gamma(TM). \tag{2.24}$$

Let $\{e_i\}_{i=1}^m$ be a geodesic frame field around $x\in M$. At x we have

$$\operatorname{div}^M \omega = \sum_{i=1}^m e_i\big[h\big(\xi\circ\varphi, \rho^2 d\varphi(e_i)\big)\big], \tag{2.25}$$

by equation (2.25), and the harmonicity condition of φ, we get

$$\begin{aligned}\operatorname{div}^M \omega &= \sum_{i=1}^m \Big[h\big(\nabla^\varphi_{e_i}(\xi\circ\varphi), \rho^2 d\varphi(e_i)\big) + h\big(\xi\circ\varphi, \nabla^\varphi_{e_i}\rho^2 d\varphi(e_i)\big)\Big]\\ &= \sum_{i=1}^m \Big[\rho^2 h\big(\nabla^\varphi_{e_i}(\xi\circ\varphi), d\varphi(e_i)\big) + 2\rho e_i(\rho) h\big(\xi\circ\varphi, d\varphi(e_i)\big)\Big],\end{aligned} \tag{2.26}$$

by the soliton equation, we find that

$$\begin{aligned}\rho^2\sum_{i=1}^m h\big(\nabla^\varphi_{e_i}(\xi\circ\varphi), d\varphi(e_i)\big) &= \lambda\rho^2\sum_{i=1}^m h\big(d\varphi(e_i), d\varphi(e_i)\big)\\ &\quad -\rho^2\sum_{i=1}^m \operatorname{Ric}^N\big(d\varphi(e_i), d\varphi(e_i)\big),\end{aligned} \tag{2.27}$$

by Young's inequality we have

$$-2\rho\sum_{i=1}^m e_i(\rho) h\big(\xi\circ\varphi, d\varphi(e_i)\big) \le \epsilon\rho^2|d\varphi|^2 + \frac{1}{\epsilon}|\operatorname{grad}^M\rho|^2|\xi\circ\varphi|^2, \tag{2.28}$$

for all $\epsilon > 0$. From (2.26), (2.27) and (2.28) we deduce the inequality

$$\begin{aligned}\lambda\rho^2|d\varphi|^2 - \rho^2\sum_{i=1}^{m}\operatorname{Ric}^N\big(d\varphi(e_i), d\varphi(e_i)\big) &- \operatorname{div}^M \omega \\ &\leq \epsilon\rho^2|d\varphi|^2 + \frac{1}{\epsilon}|\operatorname{grad}^M \rho|^2|\xi\circ\varphi|^2. \quad (2.29)\end{aligned}$$

Take $\epsilon = \lambda - \mu$, from (2.29) we obtain

$$\begin{aligned}\rho^2\Big[\mu|d\varphi|^2 - \sum_{i=1}^{m}\operatorname{Ric}^N\big(d\varphi(e_i), d\varphi(e_i)\big)\Big] &- \operatorname{div}^M \omega \\ &\leq \frac{1}{\lambda-\mu}|\operatorname{grad}^M \rho|^2|\xi\circ\varphi|^2. \quad (2.30)\end{aligned}$$

By the divergence theorem and inequality (2.30), we have

$$\begin{aligned}\int_M \rho^2\Big[\mu|d\varphi|^2 &- \sum_{i=1}^{m}\operatorname{Ric}^N\big(d\varphi(e_i), d\varphi(e_i)\big)\Big]v^g \\ &\leq \frac{1}{\lambda-\mu}\int_M |\operatorname{grad}^M \rho|^2|\xi\circ\varphi|^2 v^g. \quad (2.31)\end{aligned}$$

Consider the smooth function $\rho = \rho_R$ such that, $\rho \leq 1$ on M, $\rho = 1$ on the ball $B(p,R)$, $\rho = 0$ on $M\backslash B(p,2R)$ and $|\operatorname{grad}^M \rho| \leq \frac{2}{R}$. From inequality (2.31), we conclude that

$$\begin{aligned}\int_M \rho^2\Big[\mu|d\varphi|^2 &- \sum_{i=1}^{m}\operatorname{Ric}^N\big(d\varphi(e_i), d\varphi(e_i)\big)\Big]v^g \\ &\leq \frac{4}{(\lambda-\mu)R^2}\int_M |\xi\circ\varphi|^2 v^g, \quad (2.32)\end{aligned}$$

since $\int_M |\xi\circ\varphi|^2 v^g < \infty$, when $R \to \infty$, we obtain

$$\int_M \Big[\mu|d\varphi|^2 - \sum_{i=1}^{m}\operatorname{Ric}^N\big(d\varphi(e_i), d\varphi(e_i)\big)\Big]v^g = 0. \quad (2.33)$$

Consequently, $d\varphi(e_i) = 0$ for all $i = \overline{1,m}$ (because $\mu h - \operatorname{Ric}^N > 0$), that is φ is constant. □

If $M = N$ and $\varphi = Id_M$, from Theorem 3, we deduce:

Corollary 5. *Let (M, g, ξ, λ) be a complete non-compact nontrivial Ricci soliton with* Ric $<$ *μh for some constant $\mu < \lambda$. Then*

$$\int_M |\xi|^2 v^g = \infty.$$

2.4 Bi-harmonic maps to Ricci solitons

Theorem 4. *Let (M, g) be a compact orientable Riemannian manifold without boundary, and (N, h, ξ, λ) a nontrivial Ricci soliton with*

$$\mathrm{Ric}^N > \lambda h \quad \text{or} \quad \mathrm{Ric}^N < \lambda h.$$

Suppose that ξ is Jacobi-type vector field. Then any bi-harmonic map φ from (M, g) to (N, h) is constant.

Proof. We set

$$\eta(X) = h\big(\xi \circ \varphi, \nabla^{\varphi}_X \tau(\varphi)\big), \quad X \in \Gamma(TM). \tag{2.34}$$

Calculating in a geodesic frame field around $x \in M$ we have

$$\begin{aligned} \mathrm{div}^M \eta &= \sum_{i=1}^m e_i\big[h\big(\xi \circ \varphi, \nabla^{\varphi}_{e_i} \tau(\varphi)\big)\big] \\ &= \sum_{i=1}^m \Big[h\big(\nabla^{\varphi}_{e_i}(\xi \circ \varphi), \nabla^{\varphi}_{e_i} \tau(\varphi)\big) + h\big(\xi \circ \varphi, \nabla^{\varphi}_{e_i} \nabla^{\varphi}_{e_i} \tau(\varphi)\big)\Big], \end{aligned} \tag{2.35}$$

from equation (2.35), and the bi-harmonicity condition of φ, we get

$$\begin{aligned} \mathrm{div}^M \eta &= \sum_{i=1}^m \Big[h\big(\nabla^{\varphi}_{e_i}(\xi \circ \varphi), \nabla^{\varphi}_{e_i} \tau(\varphi)\big) \\ &\quad - h\big(R^N(\tau(\varphi), d\varphi(e_i))d\varphi(e_i), \xi \circ \varphi\big)\Big], \end{aligned} \tag{2.36}$$

the first term on the left-hand side of (2.36) is

$$\begin{aligned} \sum_{i=1}^m h\big(\nabla^{\varphi}_{e_i}(\xi \circ \varphi), \nabla^{\varphi}_{e_i} \tau(\varphi)\big) &= \sum_{i=1}^m \Big[e_i\big[h\big(\nabla^{\varphi}_{e_i}(\xi \circ \varphi), \tau(\varphi)\big)\big] \\ &\quad - h\big(\nabla^{\varphi}_{e_i} \nabla^{\varphi}_{e_i}(\xi \circ \varphi), \tau(\varphi)\big)\Big], \end{aligned} \tag{2.37}$$

by equations (2.36), (2.37), and the following property

$$h(R^N(X, Y)Z, W) = h(R^N(W, Z)Y, X),$$

where $X, Y, Z, W \in \Gamma(TN)$, we conclude that

$$\begin{aligned} \mathrm{div}^M \eta &= \mathrm{div}^M h\big(\nabla^{\varphi}_{\cdot}(\xi \circ \varphi), \tau(\varphi)\big) - \sum_{i=1}^m h\big(\nabla^{\varphi}_{e_i} \nabla^{\varphi}_{e_i}(\xi \circ \varphi), \tau(\varphi)\big) \\ &\quad - \sum_{i=1}^m h\big(R^N(\xi \circ \varphi, d\varphi(e_i))d\varphi(e_i), \tau(\varphi)\big), \end{aligned} \tag{2.38}$$

since ξ is a Jacobi-type vector field, we have

$$\mathrm{div}^M \eta = \mathrm{div}^M h\big(\nabla^{\varphi}_{\cdot}(\xi \circ \varphi), \tau(\varphi)\big) - h\big(\nabla^N_{\tau(\varphi)} \xi, \tau(\varphi)\big), \tag{2.39}$$

by the soliton equation, we get

$$\begin{aligned}\operatorname{div}^M \eta &= \operatorname{div}^M h\big(\nabla^{\varphi}_{\cdot}(\xi\circ\varphi),\tau(\varphi)\big)\\ &\quad -\lambda|\tau(\varphi)|^2+\operatorname{Ric}^N(\tau(\varphi),\tau(\varphi)),\end{aligned} \tag{2.40}$$

from equation (2.40), and the divergence theorem, with $\operatorname{Ric}^N < \lambda h$ (or $\operatorname{Ric}^N > \lambda h$), we get $\tau(\varphi) = 0$, i.e., φ is harmonic map, so by the Proposition 2, φ is constant. □

From Theorem 4, we deduce:

Corollary 6. *Let (M, g, ξ, λ) be a compact nontrivial Ricci soliton with*

$$\operatorname{Ric} > \lambda g \quad \text{or} \quad \operatorname{Ric} < \lambda g.$$

Then ξ is not Jacobi-type vector field.

Acknowledgement *The author would like to thank the editor and the reviewers for their useful remarks and suggestions.*

References

[1] P. Baird, J. C. Wood, *Harmonic morphisms between Riemannain manifolds*, Clarendon Press Oxford (2003).

[2] D. E. Blair, *Contact Manifolds in Riemannian Geometry*, Lecture Notes in Mathematics, Springer (1976).

[3] R. Caddeo, S. Montaldo, C. Oniciuc, Biharmonic submanifolds of S^3, *Internat. J. Math.* **12** (2001), 867-876.

[4] N. Course, f-harmonic maps which map the boundary of the domain to one point in the target, *New York J. Math.* **13** (2007), 423-435.

[5] S. Deshmukh, Jacobi-type vector fields and Ricci soliton, *Bull. Math. Soc. Sci. Math. Roumanie* **55 (103)** (2012), 41-50.

[6] K. L. Duggal, R. Sharma, *Symmetries of Spacetimes and Riemannian Manifolds*, Kluwer Academic Publishers, Dordrecht (1999).

[7] J. Eells, J. H. Sampson, Harmonic mappings of Riemannian manifolds, *Amer. J. Math.* **86** (1964), 109-160.

[8] R. Hamilton, The Ricci flow on surfaces, *Contemp. Math.* **71** (1988), 237-262.

[9] R. Hamilton, The Harnack estimate for the Ricci flow, *J. Differential Geom.* **37** (1993), 225-243.

[10] G. Y. Jiang, 2-harmonic maps and their first and second variational formulas, *Chinese Ann. Math. Ser. A* **7** (1986), 389-402.

[11] W. Kühnel, H. Rademacher, Conformal vector fields on pseudo-Riemannian spaces, *Differential Geometry and its Applications* **7** (1997), 237-250.

[12] E. Loubeau, C. Oniciuc, On the biharmonic and harmonic indices of the Hopf map, *Trans. Amer. Math. Soc.* **359** (2007), 5239-5256.

[13] A. Mohammed Cherif, Some results on harmonic and bi-harmonic maps, *Int. J. Geom. Methods Mod. Phys.* **14** (2017).

[14] B. O'Neil, *Semi-Riemannian Geometry*, Academic Press, New York (1983).

[15] S. Ouakkas, R. Nasri, M. Djaa, On the f-harmonic and f-biharmonic maps, *JP Journal of Geometry and Topology* **10** (2010), 11-27.

[16] S. Pigola, M. Rimoldi, A. G. Setti, Remarks on non-compact gradient Ricci solitons, *Math. Z.* **268** (2011), 777-790.

[17] Y. Xin, *Geometry of harmonic maps*, Fudan University (1996).

[18] K. Yano, T. Nagano, The de Rham decomposition, isometries and affine transformations in Riemannian space, *Jpn. J. Math.* **29** (1959), 173-184.

[19] S. T. Yau, Harmonic functions on complete Riemannian manifolds, *Comm. Pure Appl. Math.* **28** (1975), 201-228.

Received: 08.07.2023
Revised: 04.10.2023
Accepted: 08.10.2023

University Mustapha Stambouli Mascara, Faculty of Exact Sciences, Mascara 29000, Algeria
E-mail: `a.mohammedcherif@univ-mascara.dz`