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On super Catalan numbers modulo 8

by

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Abstract

In this note, we completely determine the half of the super Catalan numbers $S(m,n) = \binom{2m}{m}\binom{2n}{n}/\binom{m+n}{m}$ modulo 8, which generalizes Eu et al's congruence regarding Catalan numbers C_n modulo 8. In particular, we show that $S(m,n)/2 \not\equiv_8 7$ for all positive integers m and n. We also derive the half of central binomial coefficients $\binom{2n}{n}$ modulo 8 as a corollary.

Key Words: Catalan numbers, super Catalan numbers, central binomial coefficients.

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1 Introduction

The Catalan numbers, given by

$$C_n = \frac{1}{n+1}\binom{2n}{n}, \quad n \geq 0$$

occur in various counting problems. For instance, C_n is the number of monotonic lattice paths along the edges of a grid with $n \times n$ square cells, which do not pass above the diagonal, and is also the number of permutations of $\{1, \cdots, n\}$ that avoid the permutation pattern 123. We refer to [10] for many different combinatorial interpretations of the Catalan numbers.

In the past half-century, the divisibility of Catalan numbers has been widely discussed by several mathematicians. It is well known that C_n is odd if and only if $n = 2^k - 1$ for a nonnegative integer k. Let p be a prime. Alter and Kubota [2] showed that $p \nmid C_{p^k-1}$ for any nonnegative integer k, and that the subsequence of Catalan numbers which are divisible by p and the subsequence of Catalan numbers which are not divisible by p form into blocks.

For any positive integer n, let $\alpha(n)$ be the highest power index of base 2 such that $2^{\alpha(n)}$ divides n, and $d(n)$ be the sum of the digits in the base 2 expansion of n. It follows from Kummer's result on the order of a binomial coefficient [7] that $\alpha(C_n) = d(n+1) - 1$. It's worth mentioning that Deutsch and Sagan [4] proved this formula through a combinatorial approach. One can easily deduce from this formula that C_n is odd if and only if $n = 2^k - 1$ for some $k \in \mathbb{N}$. Deutsch and Sagan [4] also derived various interesting congruences modulo 3 for the Catalan numbers, central binomial coefficients and related sequences.

Inspired by Deutsch and Sagan's results [4], Eu, Liu and the third author [5] studied the nonzero congruences for Catalan numbers and completely determined C_n modulo 8 as follows

$$C_n \equiv_8 \begin{cases} 1 & \text{if } n = 0 \text{ or } n = 1, \\ 2 & \text{if } n = 2^{a+1} + 2^a - 1 \text{ for some } a \geq 0, \\ 4 & \text{if } n = 2^a + 2^b + 2^c - 1 \text{ for some } a > b > c \geq 0, \\ 5 & \text{if } n = 2^a - 1 \text{ for some } a \geq 2, \\ 6 & \text{if } n = 2^a + 2^b - 1 \text{ for some } a - 2 \geq b \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

From (1.1), we see that $C_n \not\equiv_8 3$ and $C_n \not\equiv_8 7$ for any nonnegative integer n. Subsequently, Xin and Xu [11] gave an alternative proof of (1.1) by using a new recursion for Catalan numbers, and further studied Catalan numbers modulo 2^r. It is worth mentioning that Krattenthaler and Müller [8] recently established some interesting congruences modulo powers of 2 for Motzkin numbers and related sequences.

The super Catalan numbers named by Gessel [6] are given by

$$S(m,n) = \frac{\binom{2m}{m}\binom{2n}{n}}{\binom{m+n}{m}},$$

for nonnegative integers m and n. Note that these numbers $S(m,n)$ are always integral and $S(1,n)/2$ coincides with the Catalan number C_n. We remark that these numbers $S(m,n)$ should not be confused with the Schröder–Hipparchus numbers, which are sometimes also called super Catalan numbers. Some interpretations of $S(m,n)$ for some special values of m have been studied by several authors (see, for example, [1, 3, 9]). It is still an open problem to find a general combinatorial interpretation for the super Catalan numbers.

The motivation of the note is to generalize Eu-Liu-Yeh congruence (1.1) and completely determine $S(m,n)/2$ modulo 8 as follows.

Theorem 1. *For nonnegative integers m and n with $m+n>0$, we have*

$$\frac{S(m,n)}{2} \equiv_8 \begin{cases} 1 & \text{if } m = n = 1 \text{ or } \{m,n\} = \{0,1\}, \\ 2 & \text{if } m+n = 2^a + 2^b \text{ for some } a-2 \geq b \geq 0 \text{ and } (m,n) \equiv_2 0 \\ & \text{or } m+n = 2^{a+1} + 2^a \text{ for some } a \geq 0 \text{ and } (m,n) \equiv_2 1, \\ 3 & \text{if } m+n = 2^a \text{ for some } a \geq 1 \text{ and } m \equiv_2 0, \\ 4 & \text{if } m+n = 2^a + 2^b + 2^c \text{ for some } a > b > c \geq 0, \\ 5 & \text{if } m+n = 2^a \text{ for some } a \geq 2 \text{ and } m \equiv_2 1, \\ 6 & \text{if } m+n = 2^a + 2^b \text{ for some } a-2 \geq b \geq 0 \text{ and } (m,n) \equiv_2 1 \\ & \text{or } m+n = 2^{a+1} + 2^a \text{ for some } a \geq 0 \text{ and } (m,n) \equiv_2 0, \\ 0 & \text{otherwise.} \end{cases} \tag{1.2}$$

Theorem 1 implies that $S(m,n)/2 \not\equiv_8 7$ for all nonnegative integers m and n with $m+n>0$. Letting $m=1$ in (1.2) reduces to (1.1). Letting $m=0$ in (1.2) leads us to the following congruence regarding the half of the central binomial coefficients $\binom{2n}{n}$.

Corollary 1. *For any positive integer n, we have*

$$\frac{1}{2}\binom{2n}{n} \equiv_8 \begin{cases} 1 & \text{if } n=1, \\ 2 & \text{if } n=3 \text{ or } n=2^a+2^b \text{ for some } a-2\geq b\geq 1, \\ 3 & \text{if } n=2^a \text{ for some } a\geq 1, \\ 4 & \text{if } n=2^a+2^b+2^c \text{ for some } a>b>c\geq 0, \\ 6 & \text{if } n=2^a+1 \text{ for some } a\geq 2 \text{ or } n=2^{a+1}+2^a \text{ for some } a\geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

From Corollary 1, we deduce that $\frac{1}{2}\binom{2n}{n} \not\equiv_8 5$ and $\frac{1}{2}\binom{2n}{n} \not\equiv_8 7$ for any positive integer n.

The rest of the paper is organized as follows. Section 2 is devoted to some notation and preliminary results. We prove Theorem 1 in Section 3.

2 Preliminary results

Following Eu et al. [5], we first recall some notation. For any positive integer n, let $[n]_2 := \langle n_r n_{r-1}\cdots n_1 n_0\rangle_2$ denote the sequence of digits representing n in base 2, i.e., $n = n_r 2^r + n_{r-1}2^{r-1}+\cdots+n_1 2+n_0$ with $n_i\in\{0,1\}$ and $n_r\neq 0$ for some integer r. For convenience, we let $n_{r+1}=n_{r+2}=\cdots=0$. We define $|\langle n_r n_{r-1}\cdots n_1 n_0\rangle_2| = n_r 2^r+n_{r-1}2^{r-1}+\cdots+n_1 2+n_0$ and $d(n)=\sum_{i\geq 0} n_i$, which counts the number of the digit 1's from n_0 to n_r. Let $r(n)$ denote the number of runs of digit 1's in $[n]_2$.

The following lemma computes the values of $\alpha(n!)$.

Lemma 1. *(See Eu et al. [5, Lemma 2.1].) For any positive integer n, we have*

$$\alpha(n!) = n - d(n). \tag{2.1}$$

For a statement S, we set $\chi(S)=1$ if S is true, otherwise $\chi(S)=0$. For positive integers $m_1, m_2, \cdots, m_k, q_1, q_2, \cdots, q_l$, we define

$$E_4\left(3, \prod_{i=1}^{k} m_i\right) := \sum_{i=1}^{k} \chi\left(\frac{m_i}{2^{\alpha(m_i)}} \equiv_4 3\right),$$

and

$$E_4\left(3, \prod_{i=1}^{k} m_i \Big/ \prod_{j=1}^{l} q_j\right) := E_4\left(3, \prod_{i=1}^{k} m_i\right) - E_4\left(3, \prod_{j=1}^{l} q_j\right).$$

In order to prove Theorem 1, we require the parity of $E_4(3, n!)$.

Lemma 2. *(See Eu et al. [5, Lemma 2.2].) For any positive integer n, we have*

$$E_4(3, n!) \equiv_2 r(n) + n_1 + n_0. \tag{2.2}$$

In the sequence $[n]_2$, let $r_1(n)$ be the number of isolated 1's, $zr(n)$ the number of runs of digit 0's, and $zr_1(n)$ the number of isolated 0's. For $t = 3, 5, 7$, we define

$$E_8\left(t, \prod_{i=1}^{k} m_i\right) := \sum_{i=1}^{k} \chi\left(\frac{m_i}{2^{\alpha(m_i)}} \equiv_8 t\right),$$

and

$$E_8\left(t, \prod_{i=1}^{k} m_i \Big/ \prod_{j=1}^{l} q_j\right) := E_8\left(t, \prod_{i=1}^{k} m_i\right) - E_8\left(t, \prod_{j=1}^{l} q_j\right),$$

where $m_1, m_2, \cdots, m_k, q_1, q_2, \cdots, q_l$ are all positive integers.

We also require the parity of $E_8(t, n!)$ for $t = 3, 5, 7$.

Lemma 3. *(See Eu et al. [5, Lemma 4.1].) For any positive integer n, we have*

$$E_8(3, n!) \equiv_2 r_1(n) + zr(n) + n_2 + n_0, \tag{2.3}$$

$$E_8(5, n!) \equiv_2 r(n) + zr_1(n) + n_2 + n_0, \tag{2.4}$$

$$E_8(7, n!) \equiv_2 r_1(n) + n_2 + n_1 + n_0. \tag{2.5}$$

Let $[n]_2 = \langle n_u n_{u-1} \cdots n_1 n_0 \rangle_2$. Then $[2n]_2 = \langle n_u n_{u-1} \cdots n_1 n_0 0 \rangle_2$. The following equations are useful in the proof of Theorem 1: $d(2n) = d(n), r(2n) = r(n), r_1(2n) = r_1(n), (2n)_0 = 0$ and $(2n)_i = n_{i-1}$ for $i \geq 1$.

3 Proof of Theorem 1

Noting that

$$S(m, n) = \frac{(2m)!(2n)!}{(m+n)!m!n!},$$

by (2.1) we have

$$\begin{aligned} \alpha\left(S(m, n)\right) &= d(m+n) + d(m) + d(n) - d(2m) - d(2n) \\ &= d(m+n). \end{aligned} \tag{3.1}$$

From (3.1), we deduce that $S(m, n)$ is even for nonnegative integers m and n with $m + n > 0$. We shall distinguish four cases to prove Theorem 1.

Case 1 $d(n+m) \geq 4$. In this case, $n + m = 2^{a_1} + 2^{a_2} + \cdots + 2^{a_t}$ for some $a_1 > a_2 > \cdots > a_t \geq 0$ and $t \geq 4$. From (3.1), we deduce that $S(m, n)/2 \equiv 0 \pmod 8$.

Case 2 $d(n+m) = 3$. In this event, $n + m = 2^a + 2^b + 2^c$ for some $a > b > c \geq 0$. By (3.1), we have $S(m, n)/8$ is always an odd integer, and so $S(m, n)/2 \equiv 4 \pmod 8$.

Case 3 $d(n+m)=2$. In this case, $m+n=2^a+2^b$ for some $a>b\geq 0$. From (3.1), we deduce that $S(m,n)/4$ is always an odd integer. By (2.2), we have

$$
\begin{aligned}
&E_4(3, S(m,n))\\
&= E_4\left(3, \frac{(2m)!(2n)!}{(m+n)!m!n!}\right)\\
&\equiv_2 r(2m)+r(2n)-r(m+n)-r(m)-r(n)\\
&+(2n)_1+(2n)_0+(2m)_1+(2m)_0-(n+m)_1-(n+m)_0-(m)_1-(m)_0-(n)_1-(n)_0\\
&\equiv_2 r(m+n)+(m+n)_1+(m+n)_0+(m)_1+(n)_1. \qquad (3.2)
\end{aligned}
$$

Observing that

$$
\begin{aligned}
|\langle\cdots 10\rangle_2|+|\langle\cdots 10\rangle_2| &= |\langle\cdots 00\rangle_2|,\\
|\langle\cdots 00\rangle_2|+|\langle\cdots 10\rangle_2| &= |\langle\cdots 10\rangle_2|,\\
|\langle\cdots 00\rangle_2|+|\langle\cdots 00\rangle_2| &= |\langle\cdots 00\rangle_2|,
\end{aligned}
$$

we get $(m+n)_1+(m+n)_0+(m)_1+(n)_1\equiv_2 0$ for $(m,n)\equiv_2 0$. On the other hand, noticing that

$$
\begin{aligned}
|\langle\cdots 01\rangle_2|+|\langle\cdots 00\rangle_2| &= |\langle\cdots 01\rangle_2|,\\
|\langle\cdots 11\rangle_2|+|\langle\cdots 00\rangle_2| &= |\langle\cdots 11\rangle_2|,\\
|\langle\cdots 01\rangle_2|+|\langle\cdots 10\rangle_2| &= |\langle\cdots 11\rangle_2|,\\
|\langle\cdots 11\rangle_2|+|\langle\cdots 10\rangle_2| &= |\langle\cdots 01\rangle_2|,\\
|\langle\cdots 01\rangle_2|+|\langle\cdots 01\rangle_2| &= |\langle\cdots 10\rangle_2|,\\
|\langle\cdots 01\rangle_2|+|\langle\cdots 11\rangle_2| &= |\langle\cdots 00\rangle_2|,\\
|\langle\cdots 11\rangle_2|+|\langle\cdots 11\rangle_2| &= |\langle\cdots 10\rangle_2|,
\end{aligned}
$$

we obtain $(m+n)_1+(m+n)_0+(m)_1+(n)_1\equiv_2 1$ for $(m,n)\equiv_2 1$.

Furthermore, we have $r(m+n)=2$ for $a-2\geq b\geq 0$ and $r(m+n)=1$ for $a-1=b\geq 0$. It follows that $E_4(3,S(m,n))\equiv_2 0$ if and only if $a-2\geq b\geq 0$ and $(m,n)\equiv_2 0$ or $a-1=b\geq 0$ and $(m,n)\equiv_2 1$, and $E_4(3,S(m,n))\equiv_2 1$ if and only if $a-2\geq b\geq 0$ and $(m,n)\equiv_2 1$ or $a-1=b\geq 0$ and $(m,n)\equiv_2 0$. Combining the above with the fact that $S(m,n)/4\equiv_4 (-1)^{E_4(3,S(m,n))}$, we get

$$
\frac{S(m,n)}{2}\equiv_8 \begin{cases} 2 & \text{if } m+n=2^a+2^b \text{ for some } a-2\geq b\geq 0 \text{ and } (m,n)\equiv_2 0\\ & \text{or } m+n=2^{a+1}+2^a \text{ for some } a\geq 0 \text{ and } (m,n)\equiv_2 1,\\ 6 & \text{if } m+n=2^a+2^b \text{ for some } a-2\geq b\geq 0 \text{ and } (m,n)\equiv_2 1\\ & \text{or } m+n=2^{a+1}+2^a \text{ for some } a\geq 0 \text{ and } (m,n)\equiv_2 0.\end{cases}
$$

Case 4 $d(n+m)=1$. In this event, $m+n=2^a$ for some $a\geq 0$ and $S(m,n)/2$ is always an odd integer.

If $a=0$, then $\{m,n\}=\{0,1\}$, and we have $S(0,1)/2=S(1,0)/2=1$.

If $a=1$, then $m=n=1$ or $\{m,n\}=\{0,2\}$, we have $S(1,1)/2=1$ and $S(0,2)/2=S(2,0)/2=3$.

If $a=2$, then $m=n=2, \{m,n\}=\{1,3\}$ or $\{m,n\}=\{0,4\}$. It is trivial to check that $S(2,2)/2=3, S(1,3)/2=S(3,1)/2=5$ and $S(0,4)/2=S(4,0)/2\equiv_8 3$.

In what follows, we assume $a\geq 3$. By (2.3), we have

$$\begin{aligned} E_8(3,S(m,n)) \equiv_2\ & r_1(2m)+r_1(2n)-r_1(m+n)-r_1(m)-r_1(n) \\ & + zr(2m)+zr(2n)-zr(m+n)-zr(m)-zr(n) \\ & + (2m)_0+(2n)_0-(m+n)_0-(m)_0-(n)_0 \\ & + (2m)_2+(2n)_2-(m+n)_2-(m)_2-(n)_2. \end{aligned} \tag{3.3}$$

Since both m and n have the same parity, we get $zr(2m)+zr(2n)-zr(m)-zr(n)\equiv_2 0$ and $(m+n)_0+(m)_0+(n)_0\equiv_2 0$. Moreover, we have $r_1(m+n)=1, zr(m+n)=1$ and $(m+n)_2=0$. It follows from (3.3) that

$$E_8(3,S(m,n))\equiv_2 (m)_2+(n)_2+(m)_1+(n)_1. \tag{3.4}$$

By (2.4), we have

$$\begin{aligned} E_8(5,S(m,n)) \equiv_2\ & r(2m)+r(2n)-r(m+n)-r(m)-r(n) \\ & + zr_1(2m)+zr_1(2n)-zr_1(m+n)-zr_1(m)-zr_1(n) \\ & + (2m)_0+(2n)_0-(m+n)_0-(m)_0-(n)_0 \\ & + (2m)_2+(2n)_2-(m+n)_2-(m)_2-(n)_2 \\ \equiv_2\ & 1+zr_1(2m)+zr_1(2n)+zr_1(m)+zr_1(n) \\ & + (m)_2+(n)_2+(m)_1+(n)_1, \end{aligned} \tag{3.5}$$

where we have used the facts that $r(m+n)=1$ and $zr_1(m+n)=0$.

Furthermore, by (2.5) we have

$$\begin{aligned} E_8(7,S(m,n)) \equiv_2\ & r_1(2m)+r_1(2n)-r_1(m+n)-r_1(m)-r_1(n) \\ & + (2m)_0+(2n)_0-(m+n)_0-(m)_0-(n)_0 \\ & + (2m)_1+(2n)_1-(m+n)_1-(m)_1-(n)_1 \\ & + (2m)_2+(2n)_2-(m+n)_2-(m)_2-(n)_2 \\ \equiv_2\ & 1+(m)_2+(n)_2. \end{aligned} \tag{3.6}$$

Since $m+n=2^a$ for $a\geq 3$, the base 2 addition $|\langle m_u m_{u-1}\cdots m_2m_1m_0\rangle_2| + |\langle n_v n_{v-1}\cdots n_2n_1n_0\rangle_2| = |\langle (m+n)_w(m+n)_{w-1}\cdots(m+n)_2(m+n)_1(m+n)_0\rangle_2|$ belongs to one of the following five cases:

$$|\langle\cdots 000\rangle_2| + |\langle\cdots 000\rangle_2| = |\langle\cdots 000\rangle_2|, \tag{3.7}$$

$$|\langle\cdots 100\rangle_2| + |\langle\cdots 100\rangle_2| = |\langle\cdots 000\rangle_2|, \tag{3.8}$$

$$|\langle\cdots 110\rangle_2| + |\langle\cdots 010\rangle_2| = |\langle\cdots 000\rangle_2|, \tag{3.9}$$

$$|\langle\cdots 001\rangle_2| + |\langle\cdots 111\rangle_2| = |\langle\cdots 000\rangle_2|, \tag{3.10}$$

$$|\langle\cdots 011\rangle_2| + |\langle\cdots 101\rangle_2| = |\langle\cdots 000\rangle_2|. \tag{3.11}$$

By using (3.4)–(3.6), we list the parity of $E_8(3, S(m,n)), E_8(5, S(m,n)), E_8(7, S(m,n))$ and the values of $S(m,n)/2$ modulo 8 with respect to the five cases (3.7)–(3.11) in the following table.

	(3.7)	(3.8)	(3.9)	(3.10)	(3.11)
$E_8(3, S(m,n))$	even	even	odd	even	even
$E_8(5, S(m,n))$	odd	odd	even	odd	odd
$E_8(7, S(m,n))$	odd	odd	even	even	even
$S(m,n)/2$ mod 8	3	3	3	5	5

Here we evaluate $S(m,n)/2$ modulo 8 based on the parity of $E_8(3, S(m,n)), E_8(5, S(m,n))$, $E_8(7, S(m,n))$, and the facts that $S(m,n)/2 \equiv_8 3^{E_8(3,S(m,n))} \times 5^{E_8(5,S(m,n))} \times 7^{E_8(7,S(m,n))}$ and $3^2 \equiv 5^2 \equiv 7^2 \equiv 1 \pmod 8$.

From the above table, we deduce that

$$\frac{S(m,n)}{2} \equiv_8 \begin{cases} 3 & \text{if } m+n = 2^a \text{ for some } a \geq 3 \text{ and } m \equiv_2 0, \\ 5 & \text{if } m+n = 2^a \text{ for some } a \geq 3 \text{ and } m \equiv_2 1. \end{cases}$$

Finally, combining Cases 1–4, we complete the proof of Theorem 1.

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