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The mass process for continuous spatial Galton-Watson non-local branching processes

by

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Abstract

The main purpose of this paper is to show that the particle counting process of a continuous spatial non-local branching process is actually a Galton-Watson process in continuous time. We do this via a general principle of transference which is perhaps useful for more situations.

Key Words: Stochastic processes, branching, Galton-Watson.
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1 Introduction

Branching processes are stochastic models used to describe populations where each individual gives rise to a random number of offspring.

Starting from the classical Galton-Watson process, which was originally introduced to model extinction probabilities in family names, the theory has been extended to include continuous time and spatial motion.

In this paper we review some basic definitions and the ingredients to describe, formulate and prove the main result.

The main purpose of this note is to show that the mass process of a spatial non-local branching process is a continuous time Galton-Watson process.

We introduce a transfer principle that connects the spatial particle process with the pure branching process, and we present the proof of the main result.

In Section 2 we present the main processes, the Galton-Watson in discrete and continuous times and the non-local branching processes with spatial motion. In Section 3, we introduce the main notation and basic results. In Section 4 we discuss the main results. In the final Section 5 we discuss some extensions.

One of the main reasons of this note is the fact that a similar property holds for branching processes in continuous time on the space of measures which is stated in [13, page 95].

2 The Galton-Watson and branching processes

2.1 The Galton-Watson process in discrete time

The classical Galton-Watson process is defined as follows. An initial individual (the parent) upon death produces a random number of children. If we denote the number of children by

Z, then

$$Z \sim \begin{pmatrix} 0 & 1 & 2 & \dots \\ p_0 & p_1 & p_2 & \dots \end{pmatrix}.$$

Each child then behaves independently and identically, giving birth to a random number of offspring with the same distribution. Let X_t be the population size at generation t. Formally, we define:

1. $X_0 = 1$,
2. $X_{t+1} = \sum_{i=1}^{X_t} Z_{t,i}$, where $Z_{t,i}$ are independent copies of Z.

Figures illustrating various aspects of the Galton-Watson process are given in Figures 1.

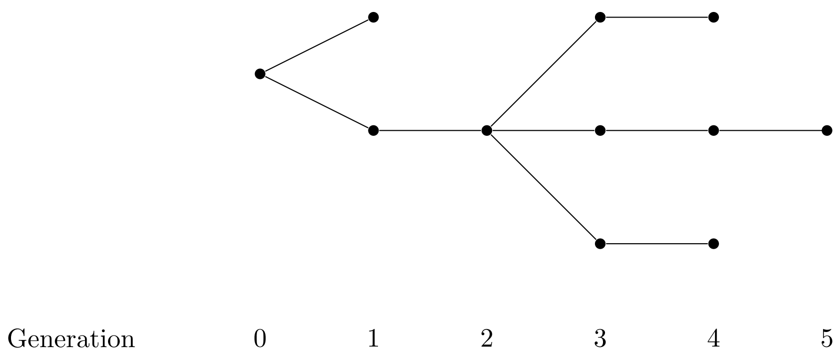

Figure 1: Galton-Watson process illustration.

2.2 Pure Galton-Watson process in continuous time

In the continuous-time version, an initial parent lives for an exponentially distributed time $Exp(a)$ and, upon death, gives birth to a random number of children with the same distribution as in the discrete model. Each child then independently lives an $Exp(a)$ time and gives birth to offspring upon death. The population at time t, denoted by X_t, evolves as a continuous-time pure branching process. For a graphical representation see Figure 2.

Pure branching processes have been recently used in [8] for solving a nonlinear Dirichlet problem (with discontinuous boundary data) related to the non-local branching processes.

2.3 Galton-Watson branching with spatial motion

In the spatial non-local branching process, in addition to the branching mechanism described above, each particle moves in space according to a Markov process. More precisely:

- An initial parent lives an exponential time $Exp(a)$ and moves in space according to a given Markov process. Upon death, it gives birth to a random number of children

$$Z \sim \begin{pmatrix} 0 & 1 & 2 & \dots \\ p_0 & p_1 & p_2 & \dots \end{pmatrix}.$$

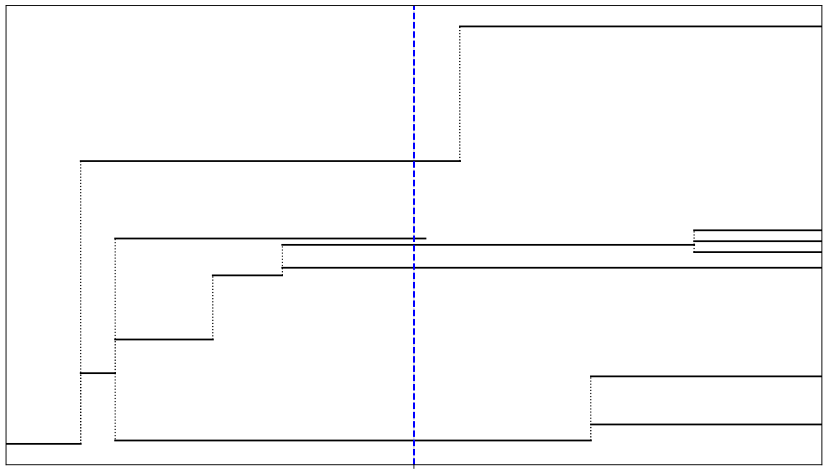

Figure 2: Representation of the time continuous pure branching process or simply Galton-Watson in continuous time. Each particle lives for an exponential amount of time and at the death gives birth to a number of off-springs, each of them following the same dynamic.

- Each child independently lives an $Exp(a)$ time and moves according to the same spatial Markov process, and gives birth to offspring upon death.
- Let $Y_{i,t}$ denote the position of the ith particle at time t, and define the particle process as
$$X_t = \sum_i \delta_{Y_{i,t}},$$
where $\delta_{Y_{i,t}}$ is the Dirac measure at $Y_{i,t}$.

A graphical image is in Figure 3.

3 Basic results and notations

Let E be a Lusin topological space (i.e., E is homeomorphic to a Borel subset of a compact metric space) denote the environmental space in which the particles move and let L be the generator of the spatial Markov process on E. We denote the sigma algebra of Borel sets by $\mathcal{B}(E)$.

For a topological space $\mathcal{X}$ and a class $\mathcal{G} \subset C(\mathcal{X})$ we denote by $\sigma(\mathcal{G})$ the sigma algebra of all Borel measurable sets of $\mathcal{X}$ of the form $G^{-1}(A)$ with $A \in \mathcal{B}(\mathbb{R})$ with $G \in \mathcal{G}$.

By semigroup on $C(U)$ we mean a family of positive operators $(P_t)_{t\geq 0} : C_b(U) \to C_b(U)$ ($C_b(U)$, the space of continuous and bounded functions) or
$$(P_t)_{t\geq 0} : C_0(U) \to C_0(U)$$
(where $C_0(U)$, the space of continuous functions vanishing at infinity) such that $P_0 = Id$ and $P_{t+s} = P_t P_s$ for $t, s \geq 0$. We say that $(P_t)_{t\geq 0}$ is C_0 if $t \to P_t f$ is continuous for each

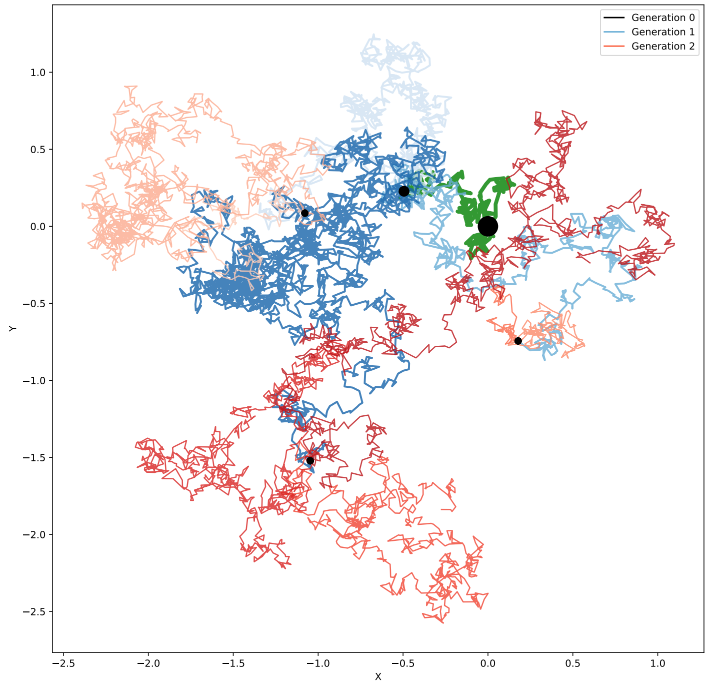

Figure 3: Branching process with spatial movement. The black dots represent the birth points of the particles. The generations are represented in the decreasing size of the particles. The colors correspond to the generation. The first generation is depicted in green, the second in blue, different nuances and the third in red.

f with respect to the uniform norm. We should point out that C_0 semigroups on $C_0(U)$ is known in the literature as *Feller* semigroups.

The generator L of the semigroup is defined in the usual way as

$$Lf = \lim_{t\to 0} \frac{P_t f - f}{t}$$

and the domain is the set of all f for which this limit exists. It is standard (see [14]) that this is a dense subspace of $C_b(U)$ (or $C_0(U)$).

The branching mechanism is given by

$$\Psi(s) = \sum_{k\geq 0} p_k s^k$$

which is the same as the generating function of the offspring distribution.

We consider the space of finite atomic measures

$$\widehat{E} = \left\{ \mu = \sum_{i=1}^{n} \delta_{x_i} : n \in \mathbb{N},\ x_i \in E \right\} \cup \{0\},$$

endowed with the topology of weak convergence.

Assume that $(P_t)_{t\geq 0}$ is the semigroup generator of a spatial Markov process. The particle process is defined as

$$X_t = \sum_i \delta_{Y_{i,t}},$$

and is characterized by the generator given by $\mathcal{L}$ (see for instance [11, Section 1.2])

$$(\mathcal{L}F_h)(\mu) = \left\langle \mu, \frac{Lh + a(\Psi(h) - h)}{h} \right\rangle F_h(\mu), \tag{1}$$

where each function $h : E \to (0,1)$ induces the test functions of the form

$$F_h(\mu) = e^{\langle \mu, \log(h) \rangle},$$

with the notation

$$\langle \mu, h \rangle := \int h \, d\mu = \sum_{i=1}^{n} h(x_i) \text{ if } \mu = \sum_{k=1}^{n} \delta_{x_i}.$$

The transition semigroup is determined by the Laplace transform via

$$\int e^{\langle \nu, \log(h) \rangle} P_t(\mu, d\nu) = e^{\langle \mu, \log(V_t h) \rangle} \tag{2}$$

or with different notations,

$$(P_t F_h)(\mu) = F_{V_t h}(\mu) \text{ for } t \geq 0 \tag{3}$$

where for $h : E \to (0,1)$, $V_t h$ is the solution of

$$\begin{cases} \frac{\partial V_t}{\partial t} = LV_t + a(\Psi(V_t) - V_t) \\ V_0 = h. \end{cases} \tag{4}$$

For references the reader can consult [11, 3, 2, 4] and for more details [13, Chapter 7].

3.1 The Case of pure branching

If we set $E = O = \{o\}$, then the process reduces to the pure branching process described earlier. In particular we can describe also this case as follows. In the first place the space $E = O$, a single point. Then functions $h : O \to (0,1)$ are completely characterized by a single constant $s \in (0,1)$. The generator L is simply $Lf = 0$ for any f. Furthermore, we also get that $\hat{O} = \mathbb{N}$ is naturally identified with the set of natural numbers. In addition, we have that for $\mu = n\delta_o$, then

$$F_s(n) = \exp(\langle \log(s), n \rangle) = s^n.$$

On the other hand, the generator of this semigroup is given by

$$(\mathcal{K}F_s)(n) = na\frac{\Psi(s) - s}{s}F_s(n) \tag{5}$$

In particular, the semigroup $(Q_t)_{t\geq 0}$ of this generator is given by

$$(Q_tF_s)(n) = F_{v_s(t)}(n) \tag{6}$$

and it must solve the Cauchy problem

$$\frac{\partial v_s}{\partial t}(t) = a(\Psi(v_s(t)) - v_s(t)) \text{ with } v_s(0) = s. \tag{7}$$

This equation has indeed a unique and well-defined solution for all $t \geq 0$ and any $s \in (0, 1)$ ([6, 5, 9]). These were detailed and thoroughly studied in [7] in a more general framework.

4 The main result

Pure branching processes have been recently used in [8] for solving a nonlinear Dirichlet problem (with discontinuous boundary data) related to the non-local branching processes.

Let the mass process be defined by

$$|X_t| := \langle X_t, 1\rangle,$$

which is the total number of particles at time t. The following result connects the spatial process to the classical pure branching process.

Theorem 4.1. *If $(X_t)_{t\geq 0}$ is generated by a C_0 semigroup (on $C_b(\widehat{E})$ or $C_0(\widehat{E})$), then the mass process $|X_t|$ is a pure Galton-Watson process in continuous time.*

One of the main ingredients of the proof is the transfer principle outlined below which is of independent interest.

Before we give the statement of the transference principle, we state a uniqueness Lemma which is the key in the proofs.

Lemma 4.2. *Assume $(P_t)_{t\geq 0}$ is a C_0 semigroup on some Banach space $\mathcal{X}$ with the generator $\mathcal{L}$ and domain $D(\mathcal{L})$. Then the Cauchy problem*

$$x'(t) = \mathcal{L}x(t) \text{ with } x(0) = x_0 \in D(\mathcal{L})$$

has a unique solution.

Proof. In the first place, if we take $y(t) = P_tx_0$, we know that (see for instance [14, Theorem 3, Section 3, Chapter IX])

$$\frac{d}{dt}P_tx_0 = \mathcal{L}P_tx_0 = P_t\mathcal{L}x_0.$$

for any $x_0 \in D(\mathcal{L})$.

Fix now some time $t > 0$ and consider

$$z(s) = P_{t-s}x(s).$$

Then,

$$z'(s) = \left(\frac{d}{ds}P_{t-s}\right)x(s) + P_{t-s}x'(s) = -P_{t-s}\mathcal{L}x(s) + P_{t-s}\mathcal{L}x(s) = 0.$$

Therefore $z(t) = z(0)$ which means that $x(t) = P_t x_0$, thus the uniqueness. ∎

The following result has some similarities to [10, Theorem 10.25 Chapter 6].

Proposition 4.3. *Assume we have a probability space* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, |)$ *and*

- *two topological spaces* U, V *and* $\phi : U \to V$ *a continuous map;*
- *a* C_0 *semigroup* $(P_t)_{t\geq 0}$ *on* $C_b(U)$ *(or* $C_0(U)$*) with generator* $\mathcal{L}$*;*
- $\mathcal{G} \circ \phi \subset D(\mathcal{L})$*;*
- *a Markov process* $(X_t)_{t\geq 0}$ *on* U *with generator* $\mathcal{L}$*;*
- *a* C_0 *semigroup* $(Q_t)_{t\geq 0}$ *on* $C_b(V)$ *(or* $C_0(V)$*) and its generator* $\mathcal{K}$ *on* V*.*
- $\mathcal{G} \subset D(\mathcal{K})$*.*

Under these assumptions, if $G \circ \phi \in D(\mathcal{L})$ *and*

$$\mathcal{L}(G \circ \phi) = \mathcal{K}(G) \circ \phi$$

for all $G \in \mathcal{G}$ *where*

$$\mathcal{G} \subset C_b(V)(or C_0(V)) \text{ such that } \sigma(\mathcal{G}) = \mathcal{B}(V) \text{ and } Q_t(\mathcal{G}) \subset \mathcal{G},\ \forall t \geq 0,$$

then the process $Y_t = \phi(X_t)$ *is a Markov process with the semigroup* $(Q_t)_{t\geq 0}$ *and generator* $\mathcal{K}$*.*

Proof. The most important observation is that

$$P_t(G \circ \phi) = (Q_t G) \circ \phi.$$

Indeed, to see that this is true, notice that

$$\frac{d}{dt}Q_t(G) \circ \phi = (\mathcal{K}Q_t(G)) \circ \phi = \mathcal{L}(Q_t(G) \circ \phi)$$

which shows that $x_t = Q_t(G) \circ \phi$ solves the equation

$$x'(t) = \mathcal{L}x(t) \text{ with } x(0) = G \circ \phi.$$

Therefore, from the uniqueness Lemma 4.2, we get that

$$Q_t(G) \circ \phi = P_t(G \circ \phi).$$

Furthermore, for any $G \in \mathcal{G}$,

$$\begin{aligned}\mathbb{E}[G(Y_t)|\mathcal{F}_s] &= \mathbb{E}[(G \circ \phi)(X_t)|\mathcal{F}_s] = P_{t-s}(G \circ \phi)(X_s) \\ &= Q_{t-s}(G)(\phi(X_s)) = Q_{t-s}(G)(Y_s)\end{aligned}$$

which, because $\sigma(\mathcal{G}) = \mathcal{B}(V)$ and monotone class theorem, extends the above relation for all continuous and bounded $G \in C_b(V)$ (or $G \in C_0(V)$), thus the conclusion. ∎

Proof of Theorem 4.1. Consider the case where $O = \{0\}$ and the spatial motion is trivial, i.e., $L \equiv 0$. On $\widehat{O}$, choose the test functions

$$G(\nu) = \exp\left(\langle \nu, \log(s) \rangle\right)$$

with the identification $h(0) = s$, where $s \in (0, 1)$.

According to the generator definition from (5), $\mathcal{K}$ on $\widehat{O}$ acts as

$$(\mathcal{K}G)(\nu) = \left\langle \nu, \frac{a(\Psi(s) - s)}{s} \right\rangle G(\nu).$$

Define the mapping $\phi : \widehat{E} \to \widehat{O}$ by

$$\phi(\mu) = \langle \mu, 1 \rangle \delta_0.$$

From (1), $F(\mu) = \exp(\langle \mu, \log(h) \rangle)$ defined by $h : E \to (0, 1)$ on $\widehat{E}$ determines the generator by

$$(\mathcal{L}F)(\mu) = \left\langle \mu, \frac{Lh + a(\Psi(h) - h)}{h} \right\rangle F(\mu).$$

Define $\mathcal{G}$ as the set of all functions of the form

$$G(\nu) = \exp(\langle \nu, \log(h) \rangle) \text{ with } h : O \to (0, 1).$$

The semigroup Q_t leaves the class $\mathcal{G}$ invariant.

It is elementary to check that $\mathcal{G}$ generates the sigma algebra of Borel sets of $\hat{O}$.

Now, for $h(o) = s$, then

$$(G \circ \phi)(\mu) = \exp\left(\langle \mu, \log(s) \rangle\right),$$

and furthermore,

$$\begin{aligned} \mathcal{L}(G \circ \phi)(\mu) &= \left\langle \mu, \frac{Ls + a(\Psi(s) - s)}{s} \right\rangle G(\phi(\mu)) \\ &= \left\langle \mu, \frac{a(\Psi(s) - s)}{s} \right\rangle G(\phi(\mu)) \\ &= \mathcal{K}(G)(\phi(\mu)). \end{aligned}$$

This verifies the hypothesis of the transfer principle and hence proves that the mass process $|X_t|$ behaves as a pure branching process. □

Remark 4.4. *The condition in Theorem 4.1 states that the proceess X_t is a C_0 process. This is true for the Galton-Watson in continuous time. The argument is based on the fact that the semigroup Q_t sends exponential functions of the form $n \to s^n$ into functions of the form $n \to v_s(t)$ (cf. (7)). It is not hard to check that $v_s(t) > 0$ for $s \in (0, 1)$ and $t > 0$, thus by Stone-Weierstrass, the set of functions $n \to s^n$ spans a dense set in the set of functions vanishing at infinity, which gurantees that Q_t is C_0 on the space $C_0(\hat{O}) = C_0(\mathbb{N})$.*

For the spacial process, we have for instance from [13, Theorem 2.5] that if E is compact and the solution $V_t h$ to (4) preserves positive functions, the process X_t is Feller, thus C_0.

However, if the semigroup V_t defined by (4) is a C_0 semigroups on $C_0(E)$, then it is not too difficult to show with the same arguments as for the Galton-Watson in continuous time that the semigroup P_t defined by (3) is also C_0 on $C_0(\widehat{E})$.

It is not clear if the Markov spatial process running the particles in space is C_0 on $C_0(E)$ then the process X_t follows to be a C_0 process as well.

The main consequence of Theorem 4.1 is the non-extinction result.

Corollary 4.5. *Let $m = \mathbb{E}[Z] = \sum_{k\geq 1} kp_k$ be the mean of the offspring distribution. Then*

- *if $m \leq 1$, then the extinction of the spatial Galton-Watson occurs with probability 1,*
- *if $m > 1$, the population has non-zero probability of non-extinction.*

In other words, the extinction of the spatial process is dictated entirely by the distribution of the offspring.

This follows from the main results relating the Galton-Watson process in continuous time to the Galton-Watson in discrete time. For instance, one such reference is [1, 12].

5 Extensions

The main result can be extended to the case of superprocesses defined on some space E and realized as processes on $\mathcal{M}(E)$, the space of measures on E. This should be contrasted to the space of point measures from $\widehat{E}$.

For superprocesses $\hat{X}$ on $\mathcal{M}(E)$, we can also look at the process $|X|_t = \langle \hat{X}_t, 1\rangle$ as a process on $[0,\infty)$. In a very similar fashion for non-local branching processes, if we take the map $\phi : \mathcal{M}(E) \to [0,\infty)$, by $\phi(\mu) = \langle \mu, 1\rangle$, and if the generators of the superprocess map into the generator of a process on $\mathcal{M}(\mathbb{R})$ we can use the same principle to show that the mass process is a branching process on $[0,\infty)$.

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