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Existence of solutions of a dynamic Signorini's problem with nonlocal friction for Viscoelastic Piezoelectric Materials

by
El Hassan Essoufi and Mostafa Kabbaj

Abstract

The aim of this paper is to study a Signorini's problem with nonlocal friction of coupled model of electro-viscoelasticity. We use a modified Kelvin-Voigt constitutive law which takes into account the piezoelectric effect of the material. After introducing the model and the functional framework, we derive a weak formulation and prove the existence of a weak solution to the problem. The proof employ the penalty method and compactness result.

Key Words: Dynamic process, viscoelastic piezoelectric, unilateral contact, nonlocal friction, Signorini problem, existence of solution.
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1 Introduction

The development of smart materials offers great potential in many structural applications and are used extensively as switches and actuary in many engineering systems, in radioelectronics, electroacoustics and measuring equipments. Piezoelectric materials are employed as both actuators and sensors in the development of these structures by taking advantage of direct and converse piezoelectric effects. General models for elastic materials with piezoelectric effects can be found in [15, 16, 18, 21, 22] and, more recently in [1, 10, 19]. Contact problems involving elastic or viscoelastic materials have received considerable attention recently in the mathematical literature, see for instance [3, 4, 8, 9, 11, 12, 14] and the references therein. Currently, there is a considerable interest in frictional contact problems involving piezoelectric materials, see for example [2, 12] and the references therein. However, there are a very few mathematical results concerning contact problems for such materials and there is a need to expand the emerging

Mathematical Theory of Contact Mechanics to include the coupling between the mechanical and electrical properties. More recently frictional contact boundary value problems with piezoelectric materials and slip dependent friction were considered in [5] in the static case. The contact with normal compliance associated to a general version of Coulomb's law of dry friction was considered in [6] in the quasistatic case.

In this paper we will study the existence of solutions of a dynamic Signorini's problem with nonlocal friction for Viscoelastic Piezoelectric Materials. We assume that the mechanical properties of the body are viscoelastic and therefore we use a modified Kelvin-Voigt constitutive law which takes into account the piezoelectric effect of the material. This work extends the recent result obtained in [4] where the analysis of a dynamic Signorini's problem with nonlocal friction for Kelvin-Voigt viscoelastic materials was provided. Indeed, with respect to the model in [4], the novelty of this paper consists in the fact that here we take into account the piezoelectric properties of the material, which leads to a new and more sophisticate mathematical model.

The paper is structured as follows. The model of the dynamic process of the viscoelastic piezoelectric body is presented in Section 2. In Section 3 we derive the weak formulation of the problem and state our main existence result, Theorem 3.1. In section 4 we present an abstract result which we use in the proof of Theorem 3.1. The section 5 is devoted to the proof of the Theorem 3.1.

2 Problem setting

We consider a linear piezoelectric body that initially occupies a bounded domain Ω in $\mathbb{R}^d$ $(d = 2, 3)$ with a smooth boundary Γ. Let $\boldsymbol{\nu} = (\nu_i)$ denote the unit outer normal on Γ and $[0, T]$ be time interval of interest, where $T > 0$. The indices i, j, k, l run between 1 and d. The summation convention over repeated indices is adopted and the index that follows a comma indicates a partial derivative with respect to the corresponding component of the independent variable.

Everywhere below we use $\mathbb{S}^d$ to denote the space of second order symmetric tensors on $\mathbb{R}^d$ while "$\cdot$" and $|\cdot|$ will represent the inner product and the Euclidean norm on $\mathbb{S}^d$ and $\mathbb{R}^d$, that is $\forall\, \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^d, \quad \forall\, \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{S}^d$

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_i v_i, \quad |\boldsymbol{v}| = (\boldsymbol{v} \cdot \boldsymbol{v})^{1/2} \quad and \quad \boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \sigma_{ij}\tau_{ij}, \quad |\boldsymbol{\tau}| = (\boldsymbol{\tau} \cdot \boldsymbol{\tau})^{1/2}.$$

We shall adopt the usual notations for normal and tangential components of displacement vector and stress :

$$\boldsymbol{u} = u_\nu \boldsymbol{\nu} + \boldsymbol{u}_\tau,\ u_\nu = u_i \nu_i,\ \boldsymbol{\sigma}_\nu = \sigma_\nu \boldsymbol{\nu} + \boldsymbol{\sigma}_\tau,\ \sigma_\nu = \sigma_{ij}\nu_i\nu_j.$$

We write the equations of motion and Coulomb's law as follows :

$$\rho\, \ddot{\boldsymbol{u}} - \operatorname{Div} \boldsymbol{\sigma} = \boldsymbol{f}_0 \qquad \text{in} \quad \Omega \times (0, T), \tag{1}$$

$$\operatorname{div} \boldsymbol{D} = q_0 \qquad \text{in} \quad \Omega \times (0, T), \tag{2}$$

Here, $\boldsymbol{\sigma}$, $\boldsymbol{u}$, $\boldsymbol{D}$ and non negative function ρ are stress tensor, the displacement vector, the dielectric displacement vector and the mass density, respectively. $\boldsymbol{f}_0$ represents the density of body forces and q_0 is the volume density of free electric charges. Notice also that Div and div represent the divergence operators for tensor and vector valued functions, that is

$$\operatorname{Div}\boldsymbol{\sigma} = (\sigma_{ij,j}) \quad \text{and} \quad \operatorname{div}\boldsymbol{D} = D_{i,i}.$$

For linear piezoelectric material we have the following constitutive relations

$$\boldsymbol{\sigma} = \mathcal{A}\varepsilon(\boldsymbol{u}) + \mathcal{B}\varepsilon(\dot{\boldsymbol{u}}) - \mathcal{E}^*\boldsymbol{E}(\varphi) \tag{3}$$

$$\boldsymbol{D} = \mathcal{E}\varepsilon(\boldsymbol{u}) + \boldsymbol{\beta}\boldsymbol{E}(\varphi) \tag{4}$$

where $\mathcal{A} = (a_{ijkl})$, $\mathcal{B} = (b_{ijkl})$, $\mathcal{E} = (e_{ijk})$ and $\boldsymbol{\beta} = (\beta_{ij})$ are respectively the (fourth-order) elasticity tensor, the (fourth-order) viscosity tensor, the (third-order) piezoelectric tensor and the electric permittivity tensor. ε and $\boldsymbol{E}$ are respectively the linearized strain tensor and the electric intensity vector. We use $\mathcal{E}^*$ to denote the transposite of the tensor $\mathcal{E}$ given by

$$\mathcal{E}\boldsymbol{\sigma}\cdot\boldsymbol{v} = \boldsymbol{\sigma}\cdot\mathcal{E}^*\boldsymbol{v} \quad \forall\,\boldsymbol{\sigma}\in\mathbb{S}^d, \quad \forall\,\boldsymbol{v}\in\mathbb{R}^d.$$

The linearized strain tensor ε and the electric intensity vector $\boldsymbol{E}$ are related to the displacement vector $\boldsymbol{u}$ and the electric potential φ through the following:

$$\varepsilon(\boldsymbol{u}) = \frac{1}{2}[\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T] \quad \text{and} \quad \boldsymbol{E} = -\nabla\varphi\cdot$$

Next, we need to prescribe the mechanical and electrical boundary conditions. To this end, we consider first a partition of Γ into three measurable parts $\Gamma_1, \Gamma_2, \Gamma_3$ such that $meas(\Gamma_1) > 0$. We assume that the body is clamped on Γ_1 and surfaces tractions of density $\boldsymbol{f}_2$ act on Γ_2 that is :

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on} \quad \Gamma_1\times(0,T) \ \text{and} \ \boldsymbol{\sigma}\boldsymbol{\nu} = \boldsymbol{f}_2 \quad \text{on} \quad \Gamma_2\times(0,T). \tag{5}$$

On Γ_3 the body is in frictional contact with an obstacle, the so-called foundation, and we model the contact with the Signorini condition and the regularized Coulomb law, that is

$$\left.\begin{array}{l} -\sigma_\nu \le 0, \quad u_\nu \le 0, \quad u_\nu\sigma_\nu = 0, \\ \left\{\begin{array}{l} |\boldsymbol{\sigma}_\tau| \le \mu|(R\sigma)_\nu| \\ |\boldsymbol{\sigma}_\tau| < \mu|(R\sigma)_\nu| \Rightarrow \dot{\boldsymbol{u}}_\tau = \boldsymbol{0} \\ |\boldsymbol{\sigma}_\tau| = \mu|(R\sigma)_\nu| \Rightarrow \exists\lambda\ge 0,\, \dot{\boldsymbol{u}}_\tau = -\lambda\boldsymbol{\sigma}_\tau \end{array}\right. \end{array}\right\} \text{on} \quad \Gamma_3\times(0,T). \tag{6}$$

Here $\mu\in L^\infty(\Omega)$, $\mu\ge 0$ is the friction coefficient and $(R\sigma)_\nu$ is a regularization of the normal contact which will be described below.

To describe the electric boundary conditions we assume that Γ is divided into two disjoints measurable parts Γ_a and Γ_b such that $meas(\Gamma_a) > 0$. We also assume, for simplicity, that the electrical potential vanishes on Γ_a and the electric charge q_2 is prescribed on Γ_b :

$$\varphi = 0 \quad \text{on} \quad \Gamma_a \times (0,T) \ \text{ and } \ \boldsymbol{D}\cdot\boldsymbol{\nu} = q_2 \quad \text{on} \quad \Gamma_b \times (0,T). \tag{7}$$

Finally, we prescribe the initials conditions that is

$$\boldsymbol{u} = \boldsymbol{u}_0, \qquad \dot{\boldsymbol{u}}(0) = \boldsymbol{v}_0 \qquad \text{in} \quad \Omega. \tag{8}$$

The classical formulation of the dynamic Signorini's problem with nonlocal friction is as follows.

Problem P. *Find a displacements field* $\boldsymbol{u} : \Omega \times [0,T] \to \mathbb{R}^d$ *and an electric potential field* $\varphi : \Omega \times [0,T] \to \mathbb{R}$ *such that* (1)-(8).

3 Weak formulation and main result

In this section we derive a weak formulation of the problem P and investigate his solvability. Everywhere in what follows we use the classical notation for the L^p and Sobolev spaces associated to Ω and Γ. Let X be a Banach space, T a positive real number then for $k = 1, 2, \ldots$, we also use de classical notations for the $W^{k,p}(0,T;X)$. We also introduce the functional spaces

$$\mathbf{H}^s(\Omega) = [H^s(\Omega)]^d, s \in \mathbb{R}, \quad \mathbf{H}^1_0(\Omega) = [H^1_0(\Omega)]^d,$$

and

$$\mathbf{L}^2(\Omega) = [L^2(\Omega)]^d, \ \ \mathbf{L}^2(\Gamma_i) = [L^2(\Gamma_i)]^d, \ \text{ for } i = 1, 2, 3.$$

Over the space $\mathbf{L}^2(\Omega)$ we use the inner product

$$(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \rho\, \boldsymbol{u}\cdot\boldsymbol{v}\, dx, \ \forall\, \boldsymbol{u}, \boldsymbol{v} \in \mathbf{L}^2(\Omega),$$

which generates an equivalent norm denoted by $|\cdot|$. Moreover, keeping in mind (5)-(6) we introduce the following space and set:

$$V = \{\boldsymbol{v} \in \mathbf{H}^1; \boldsymbol{v} = \mathbf{0} \text{ a.e. on}\, \Gamma_1\} \quad \text{and} \quad K = \{\boldsymbol{v} \in V;\, v_\nu \le 0 \text{ on } \Gamma_3\}\cdot$$

For the stress filed, we use the spaces

$$Q = \{\ \boldsymbol{\tau} = (\tau_{ij}) \mid \tau_{ij} = \tau_{ji} \in L^2(\Omega)\ \} \text{ and } Q_1 = \{\ \boldsymbol{\tau} \in Q \mid \tau_{ij,j} \in L^2(\Omega)\}.$$

Recall that the spaces Q and Q_1 are real Hilbert spaces endowed with the inner products

$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_Q = \int_\Omega \sigma_{ij}\tau_{ij}\, dx,$$

$$(\boldsymbol{\sigma},\boldsymbol{\tau})_{Q_1} = (\boldsymbol{\sigma},\boldsymbol{\tau})_Q + (\mathrm{Div}\,\boldsymbol{\sigma}, \mathrm{Div}\,\boldsymbol{\tau})_{\mathbf{L}^2(\Omega)} \ ,$$

and the associated norms $\|\cdot\|_Q$ and $\|\cdot\|_{Q_1}$, respectively. Now, over the space V we use the inner product

$$(\boldsymbol{u},\boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v}))_Q \tag{9}$$

and the associated norm $\|\cdot\|_V$. It follows from Korn's inequality (see e.g. [17] p.79) that $(V, (\cdot,\cdot)_V)$ is a real Hilbert space. Since V is dense in $\mathbf{L}^2(\Omega)$, we identify $\mathbf{L}^2(\Omega)$ and $(\mathbf{L}^2(\Omega))'$ and we write

$$V \hookrightarrow \mathbf{L}^2(\Omega) \equiv (\mathbf{L}^2(\Omega))' \hookrightarrow V'.$$

The sets $\mathbf{L}^2$ and V are Hilbert spaces with the inner products denoted by $(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle_{V,V}$, respectively. Let $\langle\cdot,\cdot\rangle_{V',V}$ denote the duality between V' and V.

Next, for the electric potential we use the functional space

$$W = \{\ \psi \in H^1(\Omega)\ ;\ \psi = 0 \text{ on } \ \Gamma_a\ \}$$

which is real Hilbert space with the inner product

$$(\varphi,\psi)_W = (\varphi,\psi)_{H^1(\Omega)},$$

and the associated norm $\|\cdot\|_W$. Since W is dense in $L^2(\Omega)$, we identify $L^2(\Omega)$ and $(L^2(\Omega))'$ and write $W \hookrightarrow L^2(\Omega) \equiv (L^2(\Omega))' \hookrightarrow W'$. We use $\langle\cdot,\cdot\rangle_{W',W}$ to denote the duality between W' and W.

Assume that the body forces and the surface tractions satisfy:

$$\boldsymbol{f}_0 \in W^{1,\infty}(0,T;\mathbf{L}^2(\Omega)) \text{ and } \boldsymbol{f}_2 \in W^{1,\infty}(0,T;\mathbf{L}^2(\Gamma_2)), \tag{10}$$

and the volume and surface densities of the electric charges satisfy:

$$q_0 \in W^{1,\infty}(0,T;L^2(\Omega)) \text{ and } q_2 \in W^{1,\infty}(0,T;L^2(\Gamma_b))\cdot \tag{11}$$

The initial conditions and the density will be supposed as follows :

$$\boldsymbol{u}_0 \in K,\ \boldsymbol{v}_0 \in V,\ \rho \in L^\infty(\Omega)\cdot \tag{12}$$

Suppose that $\mathcal{R} : Q \to Q_1$ is a linear and continuous operator which has the following properties : $(\mathcal{R}\sigma(\boldsymbol{u}_0,\boldsymbol{v}_0))_\nu = 0$ and

$$\text{if } \dot{\boldsymbol{u}}^k \to \dot{\boldsymbol{u}} \text{ in } L^2(0,T;\,\mathbf{L}^2(\Omega)) \text{ then } (\mathcal{R}\sigma^k)_\nu \to (\mathcal{R}\sigma)_\nu \text{ in } L^2(0,T;\,\mathbf{L}^2(\Gamma_3))\cdot \tag{13}$$

An operator satisfying these conditions can be obtained by extending $\boldsymbol{u}$ and $\dot{\boldsymbol{u}}$ to all of $\mathbb{R}^d$ and by using the convolution with a smooth function, which gives an averaged normal stress, see for example [11].

Next, we define the elements $\boldsymbol{f}(t) \in V'$ and $q(t) \in W'$ respectively by

$$\langle \boldsymbol{f}(t),\boldsymbol{v}\rangle_{V',V} = \int_\Omega \boldsymbol{f}_0(t)\cdot\boldsymbol{v}\,dx + \int_{\Gamma_2} \boldsymbol{f}_2(t)\cdot\boldsymbol{v}\,da$$

and

$$\langle q(t), \psi \rangle_{W',W} = \int_{\Omega} q_0(t)\psi \, dx - \int_{\Gamma_b} q_2(t)\psi \, da,$$

for all $\boldsymbol{v} \in V$, $\psi \in W$ and $t \in [0,T]$.

We denote by $a : V \times V \longrightarrow \mathbb{R}$, $b : V \times V \longrightarrow \mathbb{R}$ and $c : W \times W \longrightarrow \mathbb{R}$ the following bilinear and symmetric applications

$$a(\boldsymbol{u}, \boldsymbol{v}) := (\mathcal{A}\varepsilon(\boldsymbol{u}), \varepsilon(\boldsymbol{v}))_Q, \;\; b(\boldsymbol{u}, \boldsymbol{v}) := (\mathcal{B}\varepsilon(\boldsymbol{u}), \varepsilon(\boldsymbol{v}))_Q,$$

and

$$c(\varphi, \psi) := (\beta \nabla \varphi, \nabla \psi)_{\mathbf{L}^2(\Omega)}\cdot$$

Finally, we denote by $e : V \times W \longrightarrow \mathbb{R}$ the following bilinear application

$$e(\boldsymbol{u}, \psi) := (\mathcal{E}\varepsilon(\boldsymbol{u}), \nabla \varphi)_{\mathbf{L}^2(\Omega)} = (\mathcal{E}^* \nabla \varphi, \varepsilon(\boldsymbol{u}))_Q$$

Suppose that $\mathcal{A}$, $\mathcal{B}$ and $\boldsymbol{\beta}$ satisfy the usual properties of ellipticity and symmetry and that a_{ijkl}, b_{ijkl} and β_{ij} are in $L^\infty(\Omega)$ and that $e_{ijk} = e_{ikj} \in L^\infty(\Omega)$. It is easy to see that a and b are continuous and V - elliptic forms in the following sense: there exists $C_a > 0$, $c_a > 0$, $C_b > 0$, $c_b > 0$ such that for all $\boldsymbol{v}, \boldsymbol{w} \in V$ we have

$$|a(\boldsymbol{v}, \boldsymbol{w})| \leq C_a \|\boldsymbol{v}\|_V \|\boldsymbol{w}\|_V \;\text{ and }\; a(\boldsymbol{v}, \boldsymbol{v}) \geq c_a \|\boldsymbol{v}\|_V^2; \tag{14}$$

$$|b(\boldsymbol{v}, \boldsymbol{w})| \leq C_b \|\boldsymbol{v}\|_V \|\boldsymbol{w}\|_V \;\text{ and }\; b(\boldsymbol{v}, \boldsymbol{v}) \geq c_b \|\boldsymbol{v}\|_V^2 \cdot \tag{15}$$

Moreover, there exists m_c, M_c and c_e such that for all φ, ψ in W and $\boldsymbol{v}$ in V we have

$$|c(\varphi, \psi)| \leq M_c \|\varphi\|_W \|\psi\|_W \text{ and } c(\varphi, \varphi) \geq m_c \|\varphi\|_W^2; \tag{16}$$

and

$$|e(\boldsymbol{v}, \varphi)| \leq c_e \|\boldsymbol{v}\|_V \|\varphi\|_W \cdot \tag{17}$$

Finally, we assume that the initial conditions $\boldsymbol{u}_0$, $\boldsymbol{v}_0$ satisfy the following compatibility condition : there exist $l \in \mathbf{L}^2$ and an operator $M : V \to W$ such that

$$a(\boldsymbol{u}_0, \boldsymbol{v}) + e(\boldsymbol{v}, M\boldsymbol{u}_0) + b(\boldsymbol{v}_0, \boldsymbol{v}) = \langle f(0), \boldsymbol{v} \rangle_{V',V} + (l, \boldsymbol{v}), \; \forall\, \boldsymbol{v} \in V \cdot \tag{18}$$

We consider the following weak formulation of problem P.

Problem PW. *Find the displacement field $\boldsymbol{u} : [0,T] \to V \cap K$ and the electric potential $\varphi : [0,T] \to W$ such that $\boldsymbol{u}(0) = \boldsymbol{u}_0$, $\dot{\boldsymbol{u}}(0) = \boldsymbol{v}_0$ and*

$$\langle \dot{\boldsymbol{u}}(T), \boldsymbol{v}(T) - \boldsymbol{u}(T) \rangle_{\mathbf{H}^{-1/2}, \mathbf{H}^{1/2}} - (\boldsymbol{u}_1, \, \boldsymbol{v}(0) - \boldsymbol{u}_0) - \int_0^T (\dot{\boldsymbol{u}}, \dot{\boldsymbol{v}} - \dot{\boldsymbol{u}}) \, dt$$

$$+ \int_0^T a(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) \, dt + \int_0^T e(\boldsymbol{v} - \boldsymbol{u}, \varphi) \, dt + \int_0^T b(\dot{\boldsymbol{u}}, \boldsymbol{v} - \boldsymbol{u}) \, dt$$

$$+\int_{\Gamma_3}\mu|(\mathcal{R}\boldsymbol{\sigma})_\nu|(|\boldsymbol{v}_\tau|-|\dot{\boldsymbol{u}}_\tau-\boldsymbol{u}_\tau+\dot{\boldsymbol{u}}_\tau|-|\dot{\boldsymbol{u}}_\tau|)\,ds\geq\int_0^T\langle\boldsymbol{f},\boldsymbol{v}-\boldsymbol{u}\rangle_{V',V}\,dt, \quad (19)$$

$$\int_0^T c(\varphi,\psi)\,dt-\int_0^T e(\boldsymbol{u}),\psi)\,dt=\int_0^T\langle q,\psi\rangle_{W',W}\,dt \quad (20)$$

for all $\boldsymbol{v}\in L^\infty(0,T;V)\cap W^{1,2}(0,T;\mathbf{L}^2(\Omega))$ *with* $\boldsymbol{v}(t)\in K$, $\forall\, t\in[0,T]$ *and for all* $\psi\in L^\infty(0,T;W)\cdot$ *Where* $\langle\cdot,\cdot\rangle_{\mathbf{H}^{-1/2},\mathbf{H}^{1/2}}$ *denotes the duality pairing between* $\mathbf{H}^{-1/2}$ *and* $\mathbf{H}^{1/2}$.

The equivalence between P and PW can be easily proved by using Green's formula and an integration by parts.

Our main existence result, which we prove in section 5, is the following :

Theorem 3.1. *Under the previous notations and assumptions* (10)-(18) *there exists a solution* $(\boldsymbol{u},\varphi)$ *of Problem* PW *such that*

$$\boldsymbol{u}\in W^{1,2}(0,T;V)\cap C^1([0,T];\mathbf{H}^{-\frac{1}{2}}(\Omega)), \quad (21)$$

and

$$\varphi\in W^{1,2}(0,T;W)\cdot \quad (22)$$

□

To provide the Theorem 3.1 we need an abstract result of evolutionary variational inequality obtained in [20] which we recall in the next section and the following compactness results due to J. Simon [23] which we recall here for the convenience of the reader.

Theorem 3.2. *Let* X, U *and* Y *be the Banach spaces such that* $X\hookrightarrow U\hookrightarrow Y$ *with compact imbedding* $X\hookrightarrow U$. *Let* $\mathcal{F}$ *be bounded in* $L^p(0,T;X)$, *where* $1\leq p<\infty$, *and* $\partial\mathcal{F}/\partial t=\{\dot f,\ f\in\mathcal{F}\}$ *be bounded in* $L^1(0,T;Y)\cdot$ *Then* $\mathcal{F}$ *is relatively compact in* $L^p(0,T;U)\cdot$

Let $\mathcal{F}$ *be bounded in* $L^\infty(0,T;X)$ *and* $\partial\mathcal{F}/\partial t$ *be bounded in* $L^r(0,T;Y)$ *where* $r>1\cdot$ *Then* $\mathcal{F}$ *is relatively compact in* $C([0,T];U)\cdot$ □

4 An abstract existence and uniqueness result

To provide the Theorem 3.1 we need an abstract result of evolutionary variational inequality that we present here. Let $(H_0;|\cdot|)$ and $(V_0;\|\cdot\|)$ be two Hilbert spaces with the inner products denoted by $(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle$, respectively, such that V_0 is dense in H_0 . We set :

$$W_0=W^{1,2}(0,T;V_0)\cap W^{2,2}(0,T;H_0); \quad (23)$$

$$\mathcal{K}=\{u\in W_0,\ u(0)=u_0,\ \dot u(0)=v_0,\ \ u_0,\, v_0\in V_0\}. \quad (24)$$

Let $a, b : V_0 \times V_0 \to \mathbb{R}$ be two bilinear, symmetric, continuous and V_0 - elliptic forms in the following sense:

$$\exists\ M_a > 0 \text{ such that for all } v, w \in V_0 \times V_0,\ |a(v,w)| \leq M_a \|v\| \|w\|; \tag{25}$$

$$\exists\ M_b > 0 \text{ such that for all } v, w \in V_0 \times V_0,\ |b(v,w)| \leq M_b \|v\| \|w\|; \tag{26}$$

$$\exists\ m_a > 0 \text{ such that for all } v \in V_0,\ a(v,v) \geq m_a \|v\|^2; \tag{27}$$

$$\exists\ m_b > 0 \text{ such that for all } v \in V_0,\ b(v,v) \geq m_b \|v\|^2 \cdot \tag{28}$$

Let $\phi : [0,T] \times V_0^3 \to \mathbb{R}$ be a weakly continuous mapping such that for all $t \in [0,T]$, $(u, v, w) \in V_0^3$,

$$\phi(t,u,v,w) = \phi_1(u,v;w) + \phi_2(t,u,v,w),$$

and for all weakly convergent sequences (u_k) such that $u_k \rightharpoonup u \ in\ W_0$,

$$\liminf_{k \to +\infty} \int_0^T \phi(t, u_k(t), \dot{u}_k(t), \dot{u}_k(t)) dt \geq \int_0^T \phi(t, u(t), \dot{u}(t), \dot{u}(t)) dt \cdot \tag{29}$$

Moreover, for all $u, v \in V_0$ the mapping

$$z \mapsto \phi_1(u,v,z) \ \text{ is linear}; \tag{30}$$

$$\phi_2(0, u_0, v_0, \cdot) = 0 \ \text{ and } \ \phi_2(t,\ u,\ v,\ \cdot) \ \text{ is a semi-norm } \forall t \in [0,T], \tag{31}$$

$\exists \eta > 0$ such that $\forall\, t_1,\, t_2\, \in [0,T]$, $\forall\, u_1,\, u_2,\, v_1,\, v_2,\, w_1,\, w_2 \in V_0$,

$$|\phi(t_1,\, u_1,\, v_1,\, w_1) - \phi(t_1,\, u_1,\, v_1,\, w_2) + \phi(t_2,\, u_2,\, v_2,\, w_2) - \phi(t_2,\, u_2,\, v_2,\, w_1)|$$
$$\leq \eta(\|u_1 - u_2\| + \|v_1 - v_2\| + \|t_1 - t_2\|) \|w_1 - w_2\| \cdot \tag{32}$$

Let $f : [0,T] \to V_0$ be in $W^{1,\infty}(0,T;V_0)$. We assume that the initial conditions u_0 and v_0 verify the following compatibility condition: there exists $l \in H_0$ such that

$$a(u_0,\, w) + b(v_0,\, w) + \phi_1(u_0,\, v_0,\, w) = \langle f(0), w \rangle + (l,\, w) \ \ \forall\, w \ \in \ V_0 \cdot \tag{33}$$

We consider the following problem.

Problem Q : *Find* $u \in \mathcal{K}$ *such that for almost all* $t \in]0,T[$ such that:

$$(\ddot{u}(t), v - \dot{u}(t)) + a(u(t), v - \dot{u}(t)) + b(\dot{u}(t),\ v - \dot{u}(t))$$
$$+ \phi(t, u(t), \dot{u}(t),\ v) - \phi(t, u(t), \dot{u}(t),\ \dot{u}(t)) \geq \langle f(t), v - \dot{u}(t) \rangle \ , \forall\, v \in V_0 \ \cdot$$

Theorem 4.1. *Under the above notations and the assumptions* (23)-(33) *there exists a unique solution* u *of Problem* Q $\cdot$

The proof of this result is presented in [20].

5 Proof of Theorem 3.1

The proof of Theorem 3.1 will be carried out in several steps. The coupling system and the Signorini condition lead to considerable difficulties in the analysis of the problem. Therefore, we first consider approximate problems.

Problem P_V^ϵ. *For* $\epsilon > 0$, *find the displacement field* $\boldsymbol{u}_\epsilon : [0,T] \to V$ *and the electric potential* $\varphi_\epsilon : [0,T] \to W$ *such that* $\boldsymbol{u}_\epsilon(0) = \boldsymbol{u}_0,\ \dot{\boldsymbol{u}}_\epsilon(0) = \boldsymbol{v}_0$ *and*

$$(\ddot{\boldsymbol{u}}_\epsilon, \boldsymbol{v} - \dot{\boldsymbol{u}}_\epsilon) + a(\boldsymbol{u}_\epsilon, \boldsymbol{v} - \dot{\boldsymbol{u}}_\epsilon) + e(\boldsymbol{v} - \dot{\boldsymbol{u}}_\epsilon, \varphi_\epsilon) + b(\dot{\boldsymbol{u}}_\epsilon, \boldsymbol{v} - \dot{\boldsymbol{u}}_\epsilon)$$

$$+\frac{1}{\epsilon}\int_{\Gamma_3}[u_{\epsilon\nu}(t)]^+(v_\nu - \dot{u}_{\epsilon\nu}(t))\,ds + \int_{\Gamma_3}\mu|\mathcal{R}\boldsymbol{\sigma}_{\epsilon\nu}|(|\boldsymbol{v}_\tau| - |\dot{\boldsymbol{u}}_{\epsilon\tau}|)\,ds \geq \langle \boldsymbol{f}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}_\epsilon\rangle_{V',V},$$

$$c(\varphi_\epsilon(t), \psi) - e(\boldsymbol{u}_\epsilon(t)), \psi) = \langle q(t), \psi\rangle_{W',W}$$

$\forall\, \boldsymbol{v} \in V$ *and* $\forall\, \psi \in W,\ t \in [0,T]\cdot$

Clearly, problem P_V^ϵ represents the variational formulation of the contact problem P in which the Signorini condition in (6) was replaced by the normal compliance condition

$$\sigma_\nu = -\frac{1}{\epsilon}[u_\epsilon \nu]^+ \quad \text{on} \quad \Gamma_3\cdot$$

We establish the unique solvability of these problems and obtain the necessary a priori estimates. We first prove the following equivalence result.

Lemma 5.1. *The couple* $(\boldsymbol{u}_\epsilon, \varphi_\epsilon)$ *is a solution to problem* P_V^ϵ *if only if for all* $\boldsymbol{v} \in V$ *and* $t \in [0,T]$ *we have*

$$(\ddot{\boldsymbol{u}}_\epsilon(t), \boldsymbol{v} - \dot{\boldsymbol{u}}_\epsilon(t)) + a(\boldsymbol{u}_\epsilon(t), \boldsymbol{v} - \dot{\boldsymbol{u}}_\epsilon(t)) + b(\dot{\boldsymbol{u}}_\epsilon, \boldsymbol{v} - \dot{\boldsymbol{u}}_\epsilon) \tag{34}$$

$$+\langle \Pi^* C^{-1}\Pi \boldsymbol{u}_\epsilon(t), \boldsymbol{v} - \dot{\boldsymbol{u}}_\epsilon(t)\rangle_{V',V} + \frac{1}{\epsilon}\int_{\Gamma_3}[u_{\epsilon\nu}(t)]^+(v_\nu - \dot{u}_{\epsilon\nu}(t))\,ds$$

$$+\int_{\Gamma_3}\mu|(\mathcal{R}\boldsymbol{\sigma})_{\epsilon\nu}|(|\boldsymbol{v}_\tau| - |\dot{\boldsymbol{u}}_{\epsilon\tau}|)\,ds \geq \langle \boldsymbol{f}(t) - \Pi^* C^{-1} q(t), \boldsymbol{v} - \dot{\boldsymbol{u}}_\epsilon(t)\rangle_{V',V},$$

$$\boldsymbol{u}_\epsilon(0) = \boldsymbol{u}_0 \quad , \quad \dot{\boldsymbol{u}}_\epsilon(0) = \boldsymbol{v}_0\ , \tag{35}$$

$$C\varphi_\epsilon(t) = \Pi \boldsymbol{u}_\epsilon(t) + q(t)\ , \tag{36}$$

where $C : W \to W'$, $\Pi : V \to W'$ *and* $\Pi^* : W \to V'$ *were be defined below.* □

Proof. Let $(\boldsymbol{u}_\epsilon, \varphi_\epsilon)$ be a solution of P_V^ϵ. We solve the equation :

$$c(\varphi_\epsilon(t), \psi) = e(\boldsymbol{u}_\epsilon(t), \psi) + \langle q(t), \psi\rangle_{W',W},\ \forall\, \psi \in W, \forall\, t \in [0,T]\cdot \tag{37}$$

on $\boldsymbol{u}_\epsilon(t)$. To this end, let $\boldsymbol{u}_\epsilon : [0,T] \to V$ and find $\varphi_\epsilon : [0,T] \to W$ such that (37) holds. From (16) the symetric bilinear form $c(\varphi_\epsilon, \psi)$ is continuous and coercive on W. From (11) and (17) it is easy to see that the linear form :

$$\psi \mapsto (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_\epsilon(t)), \nabla\psi)_{\mathbf{L}^2(\Omega)} + \langle q(t), \psi\rangle_{W',W},$$

is continuous on W. Using the Lax-Milgram theorem, we conclude that the problem (37) has unique solution $\varphi_\epsilon(t) \in W$ for all $t \in [0,T]$. Moreover, by using Riesz's representation theorem we find the operators $C : W \to W'$, $\Pi : V \to W'$ and Π^* (adjoint of Π): $W \to V'$ defined by :

$$\langle C\varphi_\epsilon, \psi\rangle_{W',W} = c(\varphi_\epsilon, \psi)\ , \quad \forall\, \psi \in W \tag{38}$$

$$\langle \Pi\boldsymbol{v}, \psi\rangle_{W',W} = (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{v}), \nabla\psi)_{\mathbf{L}^2(\Omega)} = e(\boldsymbol{v}, \psi)\ , \quad \forall\, \psi \in W\ , \tag{39}$$

$$\langle \Pi^*\varphi_\epsilon, \boldsymbol{v}\rangle_{V',V} = (\mathcal{E}^*\nabla\varphi_\epsilon, \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q = e(\boldsymbol{v}, \varphi_\epsilon), \qquad \forall\, \boldsymbol{v} \in V\cdot \tag{40}$$

By using (16) it follows that C is positive definite-adjoint operator.

Hence, using the equations (38), (39) and (40), the equation (37) can be write in the form (36). By replacing (36) in (34) we prove that the problem P_V^ϵ is equivalent to : Find $\boldsymbol{u}_\epsilon : [0,T] \to V$ and $\varphi_\epsilon : [0,T] \to W$ such that (34), (35) and (36) are satisfied.□

Now, we solve the problem (34)-(35) on $\boldsymbol{u}_\epsilon$ by using similar method that in [4]. The existence of a solution of this problem can be obtained, for example, as application of existence results for set-valued pseudomonotone evolution inclusions, as developed recently in [11]. This method enables us, under less regular conditions on $\boldsymbol{v}_0$ and $\boldsymbol{f}$, to prove the existence of a solution having the following regularity : $\boldsymbol{u}_\epsilon \in W^{1,2}(0,T;V)$ and $\ddot{\boldsymbol{u}}_\epsilon \in L^2(0,T;V')$. Suitable estimates, which can be calculated if one assumes additional regularity properties of $\boldsymbol{v}_0$, $\boldsymbol{f}$ and the compatibility condition (18), ensure that $\boldsymbol{u}_\epsilon \in W^{2,2}(0,T;\mathbf{H})\cdot$

The existence and uniqueness result of this problem is given by the following theorem.

Theorem 5.1. *Under the assumptions Theorem 3.1 there exists a unique solution $\boldsymbol{u}_\epsilon$ of Problem (34)- (35) such that for every $\epsilon > 0$, we have*

$$\boldsymbol{u}_\epsilon \textit{ is bounded in} L^\infty(0,T;V) \tag{41}$$

$$\frac{[u_{\epsilon\nu}]^+}{\sqrt{\epsilon}} \textit{ is bounded in } L^\infty(0,T;L^2(\Gamma_3)), \tag{42}$$

$$\dot{\boldsymbol{u}}_\epsilon \textit{ is bounded in } L^\infty(0,T;\mathbf{L^2}(\Omega)) \cap L^2(0,T;\, V) \tag{43}$$

and

$$\ddot{\boldsymbol{u}}_\epsilon \textit{ is bounded in } L^2(0,T;\mathbf{H}^{-1}) \cdot \Box \tag{44}$$

□

Proof. To prove the above result we use the Theorem 4.1 for $V_0 = V$, $H_0 = \mathbf{L}^2(\Omega)$, $f = \boldsymbol{f} - \Pi^* C^{-1} q(t)$, the bilinear form b as in P_V^ϵ,

$$\tilde{a}(\boldsymbol{u}, \boldsymbol{v}) = a(\boldsymbol{u}, \boldsymbol{v}) + \langle \Pi^* C^{-1} \Pi \boldsymbol{u}, \boldsymbol{v} \rangle_{V', V},$$

$$\phi_1(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = \frac{1}{\epsilon} \int_{\Gamma_3} [u_\nu]^+ (w_\nu)\, ds$$

and

$$\phi_2(t, \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = \int_{\Gamma_3} \mu |(\mathcal{R}\boldsymbol{\sigma})_\nu| |\boldsymbol{w}_\tau|\, ds \text{ where } \boldsymbol{\sigma}(\boldsymbol{u}, \boldsymbol{v}) = A\varepsilon(\boldsymbol{u}) + B\varepsilon(\boldsymbol{v})\cdot$$

Under assumptions on $\mathcal{E}$ and $\mathcal{A}$, it is easy to see that $\tilde{a}$ satisfies (14) and that the assumption (25) hold. Moreover, by using (15)-(18) we prove that the assumptions (25)-(33) are satisfied. Then the existence and uniqueness of solution of the problem (34)-(35) follows from a routine application of the existence Theorem 4.1. Finally, by using this result and equation (36), we conclude that P_V^ϵ has unique solution.

We turn to obtain estimates on the solutions of Problem P_V^ϵ which will enable us to pass to the limit when ϵ tends towards 0. Firstly, Let

$$\langle \boldsymbol{F}(t), \boldsymbol{v} \rangle_{V', V} = \langle \boldsymbol{f}(t) - \Pi^* C^{-1} q(t), \boldsymbol{v} \rangle_{V', V},$$

and integrating (34) from 0 to t, for all t in $]0, T[$, we obtain for all $\boldsymbol{v} \in L^2(0, T; V)$

$$\int_0^t (\ddot{\boldsymbol{u}}_\epsilon(\theta), \boldsymbol{v} - \dot{\boldsymbol{u}}_\epsilon(\theta))\, d\theta + \int_0^t \tilde{a}(\boldsymbol{u}_\epsilon(\theta), \boldsymbol{v} - \dot{\boldsymbol{u}}_\epsilon(\theta))\, d\theta + \int_0^t b(\dot{\boldsymbol{u}}_\epsilon(\theta), \boldsymbol{v} - \dot{\boldsymbol{u}}_\epsilon(\theta))\, d\theta$$
$$+ \frac{1}{\epsilon} \int_0^t \int_{\Gamma_3} [u_{\epsilon\nu}(\theta)]^+ (v_\nu - \dot{u}_{\epsilon\nu}(\theta))\, d\theta$$
$$+ \int_0^t \int_{\Gamma_3} \mu |(\mathcal{R}\boldsymbol{\sigma})_{\epsilon\nu}| (|\boldsymbol{v}_\tau| - |\dot{\boldsymbol{u}}_{\epsilon\tau}(\theta)|)\, ds\, d\theta \geq \int_0^t \langle \boldsymbol{F}(\theta), \boldsymbol{v} - \dot{\boldsymbol{u}}_\epsilon(\theta) \rangle_{V', V}\, d\theta\cdot$$

The end of proof is now similar to that in [4] and not presented here.

In order to pass to the limit when $\epsilon \to 0$, we shall use Theorem 3.2. First, let $r = \infty$, $X = V$, $Y = \mathbf{H}$ and $\mathcal{F} = \{\boldsymbol{u}_\epsilon\}$. Since the imbedding $\mathcal{H}^s \to \mathcal{H}^r$, whit $s > r$, is compact (see [13]) then keeping in mind estimates (41)-(44), by Theorem 3.2, it follows that

$$\boldsymbol{u}_\epsilon \text{ is relatively compact in } C([0, T], \mathbf{H}^{1/2}(\Omega)) \tag{45}$$

In the same way, by choosing $r = \infty$, $X = \mathbf{H}$, $U = \mathbf{H}^{-1/2}(\Omega)$, $Y = \mathbf{H}^{-1}$ and $\mathcal{F} = \{\dot{\boldsymbol{u}}_\epsilon\}$ and using Lemma 4.1 and Theorem 3.2, we obtain

$$\dot{\boldsymbol{u}}_\epsilon \text{ is relatively compact in } C([0, T], \mathbf{H}^{-1/2}(\Omega))\cdot \tag{46}$$

Using again estimates (41)-(44) and Theorem 3.2 with : $X = V$, $U = \mathbf{H}$, $Y = \mathbf{H}^{-1}$, $p = 2$ and $\mathcal{F} = \{\dot{\boldsymbol{u}}_\epsilon\}$ it follows that

$$\dot{\boldsymbol{u}}_\epsilon \text{ is relatively compact in } L^2(0,T;\,\mathbf{H}(\Omega)) \tag{47}$$

We deduce from the estimates (41)-(44) and (45)- (47) that there exists a subsequence, still indexed by $\boldsymbol{u}_\epsilon$ and an element $\boldsymbol{u} \in C^1([0,T],\,\mathbf{H}^{-1/2}(\Omega))$ such that as $\epsilon \to 0$, the following convergences take place:

$$\boldsymbol{u}_\epsilon \to \boldsymbol{u} \text{ strongly in } C([0,T],\,\mathbf{H}^{1/2}(\Omega)), \tag{48}$$

$$\boldsymbol{u}_\epsilon \to \boldsymbol{u} \text{ weak * in } L^\infty(0,T;\,V), \tag{49}$$

$$\dot{\boldsymbol{u}}_\epsilon \to \dot{\boldsymbol{u}} \text{ strongly in } C([0,T],\,\mathbf{H}^{-1/2}(\Omega)), \tag{50}$$

$$\dot{\boldsymbol{u}}_\epsilon \to \dot{\boldsymbol{u}} \text{ strongly in } L^2(0,T;\,\mathbf{H})\cdot \tag{51}$$

$$\dot{\boldsymbol{u}}_\epsilon \to \dot{\boldsymbol{u}} \text{ weak in } L^2(0,T;\,V)\cdot \tag{52}$$

From (49) and (52) it follows that

$$\boldsymbol{u}_\epsilon \to \boldsymbol{u} \text{ weak in } W^{1,2}(0,T;\,V)\cdot \tag{53}$$

Note that, using (53) and the imbedding $W^{1,2}(0,T;V) \hookrightarrow C([0,T];V)$ we conclude that $\boldsymbol{u}_\epsilon(t)$ is bounded in V for all $t \in [0,T]\cdot$ By similar arguments as in [4] we have

$$\boldsymbol{u}_\epsilon(t) \to \boldsymbol{u}(t) \text{ in } V\ ,\ \ \forall\, t \in [0,T]\cdot \tag{54}$$

To pass to the limit in the friction term we need the following result which was proved in [4] and [11].

$$\dot{\boldsymbol{u}}_\epsilon \to \boldsymbol{u} \text{ in } L^2(0,T;\,\mathbf{L}^2(\Gamma_3))\cdot \tag{55}$$

Now, from (54) and the compactness of the imbedding $\mathbf{H}^{\frac{1}{2}(\Gamma)} \hookrightarrow \mathbf{L}^2(\Gamma)$, we have

$$u_{\epsilon\nu} \to u_\nu \text{ in } \mathbf{L}^2(\Gamma_3)\ ,\ \forall\ t \in]0,T[\ \cdot \tag{56}$$

Proof of Theorem 3.1.

First, by using assumption $\boldsymbol{u}_\epsilon(0) = \boldsymbol{u}_0$ and $\dot{\boldsymbol{u}}_\epsilon(0) = \boldsymbol{v}_0$ it is clear to see that $\boldsymbol{u}(0) = \boldsymbol{u}_0$ and $\dot{\boldsymbol{u}}(0) = \boldsymbol{v}_0$. By using the same technical that in [4] we deduce that $\boldsymbol{u}$ satisfies: for all $\boldsymbol{v} \in L^\infty(0,T;\,V) \cap W^{1,2}(0,T;\,\mathbf{L}^2(\Omega))$ with $\boldsymbol{v}(t) \in K$, $\forall\, t \in [0,T]$ we have

$$\langle \dot{\boldsymbol{u}}(T), \boldsymbol{v}(T) - \boldsymbol{u}(T)\rangle_{\mathbf{H}^{-1/2},\mathbf{H}^{1/2}} - (\boldsymbol{u}_1,\,\boldsymbol{v}(0) - \boldsymbol{u}_0) - \int_0^T (\dot{\boldsymbol{u}}, \dot{\boldsymbol{v}} - \dot{\boldsymbol{u}})\,dt$$

$$+\int_0^T a(\boldsymbol{u},\boldsymbol{v}-\boldsymbol{u})\,dt+\int_0^T \langle \Pi^* C^{-1}\Pi\boldsymbol{u}(t),\boldsymbol{v}-\dot{\boldsymbol{u}}(t)\rangle_{V',V}\,dt+\int_0^T b(\dot{\boldsymbol{u}},\boldsymbol{v}-\boldsymbol{u})\,dt$$

$$+\int_{\Gamma_3}\mu|(\mathcal{R}\boldsymbol{\sigma})_\nu|(|\boldsymbol{v}_\tau|-|\dot{\boldsymbol{u}}_\tau-\boldsymbol{u}_\tau+\dot{\boldsymbol{u}}_\tau|-|\dot{\boldsymbol{u}}_\tau|)\,ds\geq\int_0^T\langle\boldsymbol{F},\boldsymbol{v}-\boldsymbol{u}\rangle_{V',V}\,dt,$$

Now, using the above result together with (53) and Lemma 5.1, we deduce that there exists a subsequence, still indexed by φ_ϵ and an element $\varphi \in W^{1,2}(0,T;\,W)$ such that

$$\varphi_\epsilon \to \varphi \text{ weak in } W^{1,2}(0,T;\,W)\cdot$$

From this result and (53) it follows that

$$\int_0^T c(\varphi,\psi)\,dt-\int_0^T e(\boldsymbol{u},\psi)\,dt=\int_0^T\langle q,\psi\rangle_{W',W}\,dt,\quad \text{for all } \psi\in L^\infty(0,T;\,W)\cdot$$

We conclude that the couple $(\boldsymbol{u},\,\varphi)$ is a solution of the problem P.

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Université Moulay Ismail
Faculté des Sciences et Techniques,
Groupe d'Analyse Non Linéaire
52000 Boutalamine, Errachidia, Maroc
E-mail: essoufi@math.net
E-mail: mkabbaj@math.net