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A NEW PROOF OF EULER'S INRADIUS - CIRCUMRADIUS INEQUALITY

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Abstract. In this note, we give a possibly new proof of *Euler*'s inequality that in any triangle its circumradius R and its inradius r satisfy $R \geq 2r$.

Keywords: Euler's inequality.

MSC : 51M16

In 1767, *Euler* [3] analyzed and solved the construction problem of a triangle with given orthocenter, circumcenter, and incenter. The collinearity of the centroid with the orthocenter and circumcenter emerged from this analysis, together with the celebrated formula establishing the distance between the circumcenter and the incenter of the triangle.

Euler's triangle formula. *The distance d between the circumcenter and incenter of a triangle is given by $d^2 = R(R-2r)$, where R, r are the circumradius and inradius, respectively.*

An immediate consequence of this theorem is $R \geq 2r$, which is often referred to as *Euler's triangle inequality.* According to *Coxeter* [2], although this inequality had been published by *Euler* in 1767, it had appeared earlier in 1746 in a publication by *William Chapple.* This ubiquitous inequality occurs in the literature in many different equivalent forms [1, 4]. For example:

$$1 \geq 8 \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2},$$

$$r_a + r_b + r_c \geq 9r,$$

$$abc \geq (a+b-c)(b+c-a)(c+a-b),$$

$$(y+z)(z+x)(x+y) \geq 8xyz,$$

where a, b, c, A, B, C, r_a, r_b, r_c denote the sides, angles, exradii of the triangle, and x, y, z are arbitrary nonnegative numbers such that $a = y+z$, $b = x+z$, $c = x+y$. A proof for the last equivalent form follows immediately from the product of the three obvious inequalities

$$y+z \geq 2\sqrt{yz}, \quad z+x \geq 2\sqrt{zx}, \quad x+y \geq 2\sqrt{xy}.$$

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Many other simple approaches are known. Here we give a less obvious approach, by making use of inversion with respect to the incircle.

Theorem (Euler's triangle inequality). *In any triangle ABC, with circumradius R and inradius r, we have that*:

$$R \geq 2r.$$

Proof. Let ρ be an arbitrary line passing through the incenter I of triangle ABC and let M, N be its intersections with the circumcircle. Since $MN \leq 2R$, with equality if and only if the circumcenter O of ABC lies on MN, note that it suffices to prove that $MN \geq 4r$.

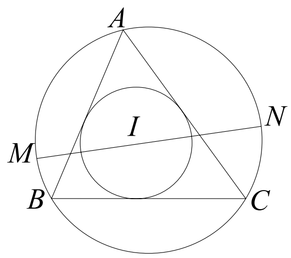

Figure 1

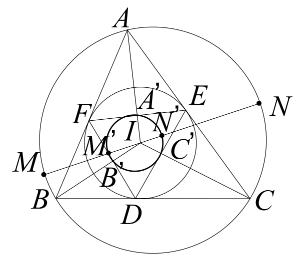

Figure 2

Denote by D, E, F the tangency points of the incircle with the sides BC, CA and AB respectively, and let A', B', C' be the midpoints of the segments EF, FD and DE.

Since:

$$IA \cdot IA' = \left(\frac{r}{\sin\dfrac{A}{2}}\right) \cdot \left(r \sin\frac{A}{2}\right) = r^2,$$

and, similarly, $IB \cdot IB' = r^2$, $IC \cdot IC' = r^2$, the points A', B', C' are the images of the vertices A, B, C under the inversion Ψ with pole I and power r^2. In this case, the image of the circumcircle (O) of ABC under Ψ is the circumcircle (O') of triangle $A'B'C'$. Since ρ passes through I, this line remains invariant under the inversion and, therefore, the images M', N' of the points M respectively N are the intersections of ρ with (O') (see Figure 2).

We thus have:

$$IM = \frac{r^2}{IM'}, \quad IN = \frac{r^2}{IN'},$$

and therefore:

$$MN = IM + IN = r^2 \cdot \left(\frac{1}{IM'} + \frac{1}{IN'}\right) \geq \frac{4}{IM' + IN'} r^2 = \frac{4r^2}{M'N'}.$$

On the other hand, since $A'B'C'$ is the medial triangle of DEF, its circumradius is $\dfrac{r}{2}$, and I is its orthocenter. It now follows that $M'N' \leq r$, hence we conclude that:

$$MN \geq \frac{4r^2}{M'N'} \geq 4r.$$

The above inequality, combined with $MN \leq 2R$, yields *Euler*'s triangle inequality.

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ASUPRA UNEI PROBLEME DE LA O. N. M. 2008

de Artur Bălăucă[1)]

Abstract. This note shows a simpler proof of a problem submited in the National Olympiad

Keywords: rectangle divided into equal squres.

MSC : 11A51

La a 59-a Olimpiadă Naţională de Matematică, desfăşurată la Timişoara în perioada 29 aprilie - 4 mai 2008, prof. *Marius Perianu* a propus următoarea problemă la clasa a VII-a:

Un dreptunghi se poate împărţi, ducând paralele la laturile sale, în 200 *de pătrate congruente şi în* 288 *de pătrate congruente. Arătaţi că dreptunghiul se poate împărţi şi în* 392 *de pătrate congruente.*

O soluţie a acestei probleme poate fi găsită în [1]. Scopul acestei note este de a prezenta o soluţie mai simplă, precum şi o generalizare.

Dacă dreptunghiul poate fi împărţit în 200 de pătrate congruente, atunci lungimea sa poate fi împărţită în a părţi congruente iar lăţimea în b părţi congruente (lungimea unei părţi în ambele cazuri fiind aceeaşi), astfel încât $a \cdot b = 200$, unde $a, b \in \mathbb{N}^*$. Cum $a > b$, rezultă că:

$$\frac{b}{a} \in \left\{\frac{1}{200}; \frac{2}{100}; \frac{4}{50}; \frac{5}{40}; \frac{8}{25}; \frac{10}{20}\right\} = A.$$

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